Further Mathematics SL P1

2010 May

School Level 12th IB Diploma Programme

Board Exam

International Baccalaureate (IB Board)

Solved

M10/5/FURMA/SP1/ENG/TZ0/XX

22107101

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 1

Thursday 20 May 2010 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

2210-7101

4 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. All students should therefore be advised to show their working.

1. *[Maximum mark: 7]*

A university Mathematics Department admits students on the basis of performance in an entrance examination which is graded 'excellent', 'very good' or 'good'. The students sit their final examination three years later when they are awarded 'first class', 'second class' or 'third class' degrees. The results for a particular group of students are summarised in the following table.

	First class	Second class	Third class
Excellent	30	14	6
Very good	8	12	5
Good	5	6	14

Stating your hypotheses, use an appropriate test to investigate at the 1 % level of significance whether or not there is an association between performance in the entrance examination and performance in the final examination. Justify your answer.

2. *[Maximum mark: 10]*

Let S be the set of matrices given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in \mathbb{R}, ad - bc = 1.$$

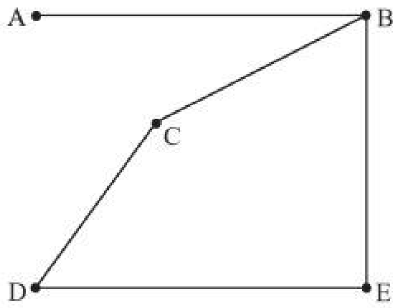
The relation R is defined on S as follows. Given $\boldsymbol{A}$, $\boldsymbol{B} \in S$, $\boldsymbol{ARB}$ if and only if there exists $\boldsymbol{X} \in S$ such that $\boldsymbol{A} = \boldsymbol{BX}$.

(a) Show that R is an equivalence relation. *[8 marks]*

(b) The relationship between a, b, c and d is changed to $ad - bc = n$. State, with a reason, whether or not there are any non-zero values of n, other than 1, for which R is an equivalence relation. *[2 marks]*

3. *[Maximum mark: 11]*

The figure below shows the graph G.



(a) (i) Write down the adjacency matrix for G.

(ii) Find the number of walks of length 4 beginning and ending at B. *[5 marks]*

(b) (i) Draw G', the complement of G.

(ii) Write down the degrees of all the vertices of G and all the vertices of G'.

(iii) Hence, or otherwise, determine whether or not G and G' are isomorphic. *[6 marks]*

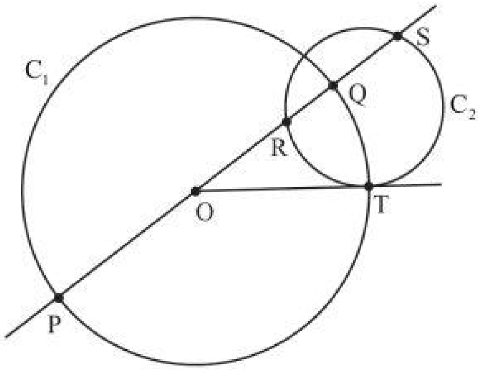
4. *[Maximum mark: 9]*

Given that $n^2 + 2n + 3 \equiv N \pmod 8$, where $n \in \mathbb{Z}^+$ and $0 \le N \le 7$, prove that N can take one of only three possible values.

5. *[Maximum mark: 11]*

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y \tan x = \sin x$, and $y = 0$ when $x = \frac{\pi}{3}$, find the maximum value of y.

6. *[Maximum mark: 12]*



The figure shows a circle C_1 with centre O and diameter [PQ] and a circle C_2 which intersects (PQ) at the points R and S. T is one point of intersection of the two circles and (OT) is a tangent to C_2.

(a) Show that $\frac{OR}{OT} = \frac{OT}{OS}$. *[2 marks]*

(b) (i) Show that $PR - RQ = 2OR$.

(ii) Show that $\frac{PR - RQ}{PR + RQ} = \frac{PS - SQ}{PS + SQ}$. *[6 marks]*

(c) Deduce that P, R, Q, S form a harmonic ratio. *[4 marks]*

M10/5/FURMA/SP1/ENG/TZ0/XX/M+

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MARKSCHEME

May 2010

FURTHER MATHEMATICS

Standard Level

Paper 1

9 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc.*
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) are often dependent on the preceding ***M*** mark.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.* do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write –1(**MR**) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled **(d)** and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc.*
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = 2\cos(5x-3)\ \ 5 \quad = 10\cos(5x-3) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $2\cos(5x-3)\ \ 5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Candidates should be penalized ***once only IN THE PAPER*** *for an accuracy error* ***(AP)***. *Award the marks as usual then write* ***(AP)*** *against the answer. On the* ***front*** *cover write –1****(AP)***. *Deduct 1 mark from the total for the paper, not the question.*

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

1. H_0: There is no association between performance in the entrance examination and performance in the final examination.
H_1: There is an association. *A1*

EITHER

p-value $= 1.19 \times 10^{-4}$ *A4*

accept H_1 because $1.19 \times 10^{-4} < 0.01$ *A2*

OR

the expected frequencies are

	First class	**Second class**	**Third class**
Excellent	21.5	16	12.5
Very good	10.75	8	6.25
Good	10.75	8	6.25

M1A1

$\chi^2_{calc} = 23.1$ *A1*

$\chi^2_{crit} = 13.277$ *A1*

accept H_1 because $23.1 > 13.277$ *A1R1*

[7 marks]

2. (a) since $\boldsymbol{A} = \boldsymbol{AI}$ where $\boldsymbol{I}$ is the identity *A1*

and $\det(\boldsymbol{I}) = 1$, *A1*

R is reflexive

$\boldsymbol{A}R\boldsymbol{B} \Rightarrow \boldsymbol{A} = \boldsymbol{BX}$ where $\det(\boldsymbol{X}) = 1$ *M1*

it follows that $\boldsymbol{B} = \boldsymbol{AX}^{-1}$ *A1*

and $\det(\boldsymbol{X}^{-1}) = \det(\boldsymbol{X})^{-1} = 1$ *A1*

R is symmetric

$\boldsymbol{A}R\boldsymbol{B}$ and $\boldsymbol{B}R\boldsymbol{C} \Rightarrow \boldsymbol{A} = \boldsymbol{BX}$ and $\boldsymbol{B} = \boldsymbol{CY}$ where $\det(\boldsymbol{X}) = \det(\boldsymbol{Y}) = 1$ *M1*

it follows that $\boldsymbol{A} = \boldsymbol{CYX}$ *A1*

$\det(\boldsymbol{YX}) = \det(\boldsymbol{Y})\det(\boldsymbol{X}) = 1$ *A1*

R is transitive

hence R is an equivalence relation *AG*

[8 marks]

(b) for reflexivity, we require $\boldsymbol{A}R\boldsymbol{A}$ so that $\boldsymbol{A} = \boldsymbol{AI}$ (for all $\boldsymbol{A} \in S$) *M1*

since $\det(\boldsymbol{I}) = 1$ and we require $\boldsymbol{I} \in S$ the only possibility is $n = 1$ *A1*

[2 marks]

Total [10 marks]

3. (a) (i) the adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

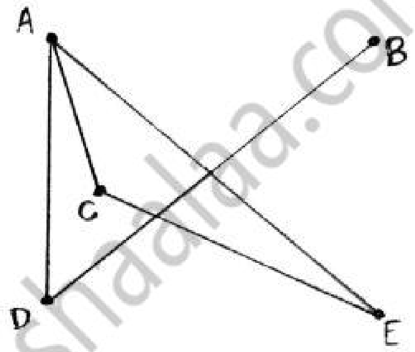
A2

Note: Award ***A2*** for correct matrix, ***A1*** for one or two errors and ***A0*** for more than two errors.

(ii) the required number of walks is the (B, B) element in the fourth power of the adjacency matrix ***(M1)***
using a GDC gives the answer as 13 ***A2***

[5 marks]

(b) (i)



A2

(ii)

	G	***G′***
A	1	3
B	3	1
C	2	2
D	2	2
E	2	2

A1A1

(iii) In G, the vertex of degree 1 is adjacent to the vertex of degree 3, whereas in G' the vertex of degree 1 is adjacent to a vertex of degree 2. They are not therefore isomorphic. ***A1R1***

Note: Accept alternative correct solutions.

[6 marks]

Total [11 marks]

4. consider the following

n	$(n^2+2n+3)(\bmod 8)$
1	6
2	3
3	2
4	3
5	6
6	3
7	2
8	3

M1A1A1

we see that the only possible values so far are 2, 3 and 6 ***R1***
also, the table suggests that these values repeat themselves but we have to prove this
let $f(n)=n^2+2n+3$, consider

$f(n+4)-f(n)=(n+4)^2+2(n+4)+3-n^2-2n-3$ ***M1***

$=8n+24$ ***A1***

since $8n+24$ is divisible by 8, ***M1***

$f(n+4)\equiv f(n)(\bmod 8)$ ***A1***

this confirms that the values do repeat every 4 values of n so that 2, 3 and 6 are the only values taken for all values of n ***R1***

[9 marks]

5. integrating factor $=e^{\int 2\tan x\,dx}$ ***M1***

$=e^{2\ln\sec x}$ ***A1***

$=\sec^2 x$ ***A1***

it follows that

$y\sec^2 x=\int \sin x\sec^2 x\,dx$ ***M1***

$=\int \sec x\tan x\,dx$ ***(A1)***

$=\sec x+C$ ***A1***

substituting,

$0=2+C$ so $C=-2$ ***M1A1***

the solution is

$y=\cos x-2\cos^2 x$ ***A1***

EITHER

using a GDC
maximum value of y is 0.125 ***A2***

OR

$y'=-\sin x+4\sin x\cos x=0$ ***M1***

$\Rightarrow \cos x=\frac{1}{4}$ (or $\sin x=0$ which leads to a minimum)

$\Rightarrow y=\frac{1}{8}$ ***A1***

[11 marks]

6. (a) by the tangent – secant theorem, *M1*

$\mathrm{OT}^2 = \mathrm{OR} \bullet \mathrm{OS}$ *A1*

so that $\dfrac{\mathrm{OR}}{\mathrm{OT}} = \dfrac{\mathrm{OT}}{\mathrm{OS}}$ *AG*

[2 marks]

(b) (i) $\mathrm{PR} - \mathrm{RQ} = \mathrm{PO} + \mathrm{OR} - (\mathrm{OQ} - \mathrm{OR})$ *A1*

$= 2\mathrm{OR}$ *AG*

(ii) attempt to continue the process set up in (b)(i) *(M1)*

$\mathrm{PR} + \mathrm{RQ} = \mathrm{PO} + \mathrm{OR} + \mathrm{OQ} - \mathrm{OR} = 2\mathrm{OT}$ *A1*

$\mathrm{PS} - \mathrm{SQ} = \mathrm{PQ} + \mathrm{QS} - \mathrm{SQ} = 2\mathrm{OT}$ *A1*

$\mathrm{PS} + \mathrm{SQ} = \mathrm{PO} + \mathrm{OS} + \mathrm{OS} - \mathrm{OQ} = 2\mathrm{OS}$ *A1*

it now follows that

$\dfrac{\mathrm{PR} - \mathrm{RQ}}{\mathrm{PR} + \mathrm{RQ}} = \dfrac{\mathrm{OR}}{\mathrm{OT}}$ and $\dfrac{\mathrm{PS} - \mathrm{SQ}}{\mathrm{PS} + \mathrm{SQ}} = \dfrac{\mathrm{OT}}{\mathrm{OS}}$ so using the result in part (a) *R1*

$\dfrac{\mathrm{PR} - \mathrm{RQ}}{\mathrm{PR} + \mathrm{RQ}} = \dfrac{\mathrm{PS} - \mathrm{SQ}}{\mathrm{PS} + \mathrm{SQ}}$ *AG*

[6 marks]

(c) cross multiplying,

$\mathrm{PR} \bullet \mathrm{PS} + \mathrm{PR} \bullet \mathrm{SQ} - \mathrm{RQ} \bullet \mathrm{PS} - \mathrm{RQ} \bullet \mathrm{SQ} = \mathrm{PR} \bullet \mathrm{PS} - \mathrm{PR} \bullet \mathrm{SQ} + \mathrm{RQ} \bullet \mathrm{PS} - \mathrm{RQ} \bullet \mathrm{SQ}$ *M1A1*

$2\mathrm{PR} \bullet \mathrm{SQ} = 2\mathrm{RQ} \bullet \mathrm{PS}$ *A1*

$\dfrac{\mathrm{PR}}{\mathrm{RQ}} = \dfrac{\mathrm{PS}}{\mathrm{SQ}}$ *A1*

hence P, R, Q, S form a harmonic ratio *AG*

[4 marks]

Total [12 marks]