Further Mathematics SL P2

2010 May

School Level 12th IB Diploma Programme

Board Exam

International Baccalaureate (IB Board)

Solved

M10/5/FURMA/SP2/ENG/TZ0/XX

22107102

International Baccalaureate®
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FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 2

Friday 21 May 2010 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

2210-7102

6 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. All students should therefore be advised to show their working.

1. *[Maximum mark: 32]*

The binary operator $*$ is defined for $a, b \in \mathbb{R}$ by $a * b = a + b - ab$.

(a) (i) Show that $*$ is associative.

(ii) Find the identity element.

(iii) Find the inverse of $a \in \mathbb{R}$, showing that the inverse exists for all values of a except one value which should be identified.

(iv) Solve the equation $x * x = 1$. *[15 marks]*

(b) The domain of $*$ is now reduced to $S = \{0, 2, 3, 4, 5, 6\}$ and the arithmetic is carried out modulo 7.

(i) Copy and complete the following Cayley table for $\{S, *\}$.

$*$	**0**	**2**	**3**	**4**	**5**	**6**
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3					
4	4					
5	5					
6	6					

(ii) Show that $\{S, *\}$ is a group.

(iii) Determine the order of each element in S and state, with a reason, whether or not $\{S, *\}$ is cyclic.

(iv) Determine all the proper subgroups of $\{S, *\}$ and explain how your results illustrate Lagrange's theorem.

(v) Solve the equation $2 * x * x = 5$. *[17 marks]*

2. *[Total mark: 16]*

Part A *[Maximum mark: 9]*

The points D, E, F lie on the sides [BC], [CA], [AB] of the triangle ABC and [AD], [BE], [CF] intersect at the point G. You are given that $CD = 2BD$ and $AG = 2GD$.

(a) By considering (BE) as a transversal to the triangle ACD, show that

$$\frac{CE}{EA} = \frac{3}{2}.$$

[2 marks]

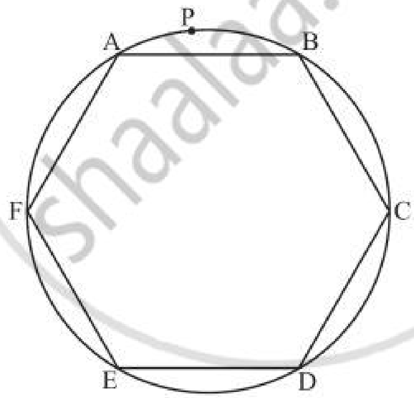
(b) Determine the ratios

(i) $\frac{AF}{FB}$;

(ii) $\frac{BG}{GE}$.

[7 marks]

Part B *[Maximum mark: 7]*

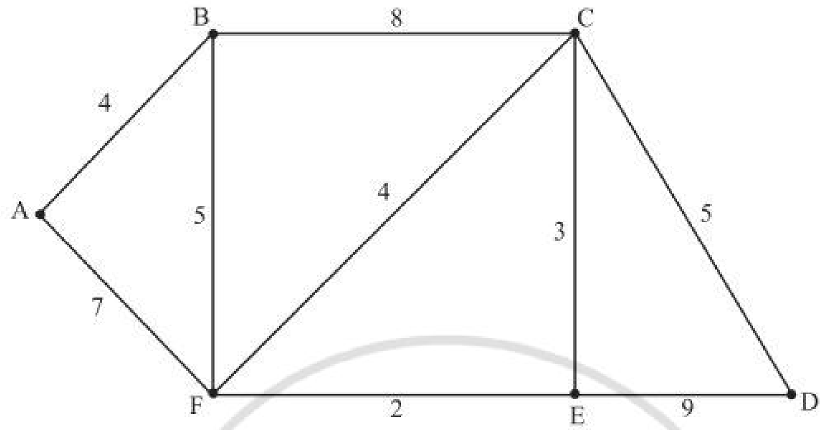


The diagram shows a hexagon ABCDEF inscribed in a circle. All the sides of the hexagon are equal in length. The point P lies on the minor arc AB of the circle. Using Ptolemy's theorem, show that

$$PE + PD = PA + PB + PC + PF.$$

3. *[Maximum mark: 18]*

The following diagram shows a weighted graph G.



(a) (i) Explain briefly what features of the graph enable you to state that G has an Eulerian trail but does not have an Eulerian circuit.

(ii) Write down an Eulerian trail in G. *[3 marks]*

(b) (i) Use Kruskal's algorithm to find and draw the minimum spanning tree for G. Your solution should indicate the order in which the edges are added.

(ii) State the weight of the minimum spanning tree. *[5 marks]*

(c) Use Dijkstra's algorithm to find the path of minimum total weight joining A to D, and state its weight. Your solution should indicate clearly the use of this algorithm. *[10 marks]*

4. *[Maximum mark: 13]*

(a) The weights, X grams, of tomatoes may be assumed to be normally distributed with mean μ grams and standard deviation σ grams. Barry weighs 21 tomatoes selected at random and calculates the following statistics.

$$\sum x = 1071; \sum x^2 = 54705$$

(i) Determine unbiased estimates of μ and σ^2.

(ii) Determine a 95 % confidence interval for μ. *[8 marks]*

(b) The random variable Y has variance σ^2, where $\sigma^2 > 0$. A random sample of n observations of Y is taken and S_{n-1}^2 denotes the unbiased estimator for σ^2. By considering the expression

$$\mathrm{Var}(S_{n-1}) = \mathrm{E}(S_{n-1}^2) - \{\mathrm{E}(S_{n-1})\}^2,$$

show that S_{n-1} is not an unbiased estimator for σ. *[5 marks]*

5. *[Maximum mark: 19]*

After a shop opens at 09:00 the number of customers arriving in any interval of duration t minutes follows a Poisson distribution with mean $\frac{t}{10}$.

(a) (i) Find the probability that exactly five customers arrive before 10:00.

(ii) Given that exactly five customers arrive before 10:00, find the probability that exactly two customers arrive before 09:30. *[7 marks]*

(b) Let the second customer arrive at T minutes after 09:00.

(i) Show that, for $t > 0$,

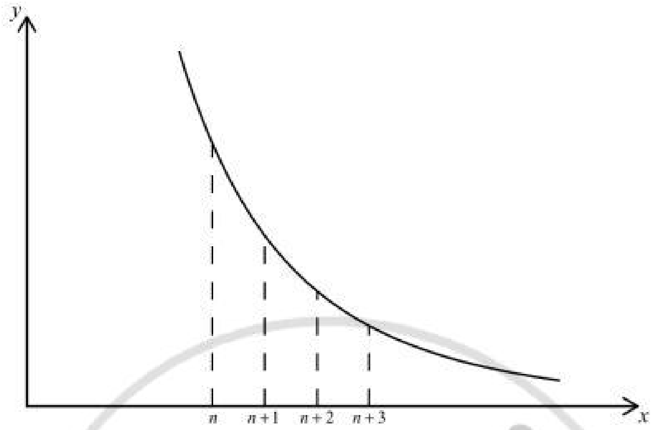
$$\mathrm{P}(T > t) = \left(1 + \frac{t}{10}\right)\mathrm{e}^{-\frac{t}{10}}.$$

(ii) Hence find in simplified form the probability density function of T.

(iii) Evaluate $\mathrm{E}(T)$.
(You may assume that, for $n \in \mathbb{Z}^+$ and $a > 0$, $\lim_{t\to\infty} t^n \mathrm{e}^{-at} = 0$.) *[12 marks]*

6. *[Maximum mark: 22]*

(a) The diagram shows a sketch of the graph of $y = x^{-4}$ for $x > 0$.



By considering this sketch, show that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=n+1}^{\infty} \frac{1}{r^4} < \int_n^{\infty} \frac{dx}{x^4} < \sum_{r=n}^{\infty} \frac{1}{r^4}.$$

[5 marks]

(b) Let $S = \sum_{r=1}^{\infty} \frac{1}{r^4}$.

Use the result in (a) to show that, for $n \geq 2$, the value of S lies between

$$\sum_{r=1}^{n-1} \frac{1}{r^4} + \frac{1}{3n^3} \text{ and } \sum_{r=1}^{n} \frac{1}{r^4} + \frac{1}{3n^3}.$$

[8 marks]

(c) (i) Show that, by taking $n = 8$, the value of S can be deduced correct to three decimal places and state this value.

(ii) The exact value of S is known to be $\frac{\pi^4}{N}$ where $N \in \mathbb{Z}^+$. Determine the value of N. *[6 marks]*

(d) Now let $T = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^4}$.

Find the value of T correct to three decimal places. *[3 marks]*

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MARKSCHEME

May 2010

FURTHER MATHEMATICS

Standard Level

Paper 2

13 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) are often dependent on the preceding ***M*** mark.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.* do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write –1(**MR**) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled **(d)** and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc.*
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = 2\cos(5x-3)\ 5 \quad = 10\cos(5x-3)$ ***A1***

Award ***A1*** for $2\cos(5x-3)\ 5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

*Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**. Award the marks as usual then write **(AP)** against the answer. On the **front** cover write –1**(AP)**. Deduct 1 mark from the total for the paper, not the question.*

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

1. (a) (i) $a*(b*c) = a*(b+c-bc)$ *M1*

$= a+b+c-bc-a(b+c-bc)$ *A1*

$= a+b+c-bc-ca-ab+abc$ *A1*

$(a*b)*c = (a+b-ab)*c$ *M1*

$= a+b-ab+c-(a+b-ab)c$ *A1*

$= a+b+c-bc-ca-ab+abc$, hence associative *AG*

(ii) let e be the identity element, so that $a*e=a$ *(M1)*

then,

$a+e-ae=a$ *A1*

$e(1-a)=0$

$e=0$ *A1*

(iii) let a^{-1} be the inverse of a, so that $a*a^{-1}=0$ *(M1)*

then,

$a+a^{-1}-aa^{-1}=0$ *A1*

$a^{-1}=\dfrac{a}{a-1}$ *A1*

this gives an inverse for all elements except 1 which has no inverse *R1*

(iv) $2x-x^2=1$ *M1*

$(x-1)^2=0$ *(A1)*

$x=1$ *A1*

[15 marks]

(b) (i)

*	**0**	**2**	**3**	**4**	**5**	**6**
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3	6	4	2	0	5
4	4	5	2	6	3	0
5	5	4	0	3	6	2
6	6	3	5	0	2	4

A3

Note: Award ***A3*** for correct table, ***A2*** for one error, ***A1*** for two errors and ***A0*** for more than two errors.

(ii) there are no new elements in the table so it is closed *A1*

there is an identity element, 0 *A1*

every row (column) has a 0 so every element has an inverse *A1*

associativity has been proved earlier *A1*

therefore $\{S, *\}$ is a group *AG*

continued ...

Question 1 continued

(iii)

Element	Order
0	1
2	2
3	6
4	3
5	6
6	3

A3

Note: Award ***A3*** for correct table, ***A2*** for one error, ***A1*** for two errors and ***A0*** for more than two errors.

it is cyclic because there are elements of order 6 ***R1***

(iv) the proper subgroups are $\{0, 2\}$, $\{0, 4, 6\}$ ***A1A1***

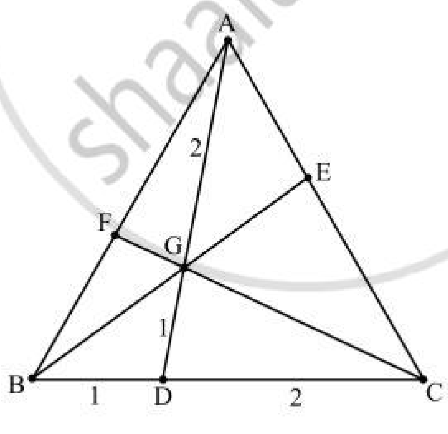
the orders of the subgroups $(2, 3)$ are factors of the order of the group (6) ***A1***

(v) recognizing $x * x = 4$ ***(M1)***
$x = 3, 6$ ***A1A1***

[17 marks]

Total [32 marks]

2. Part A



(a) using Menelaus' theorem in ΔACD,

$$\frac{\text{CE}}{\text{EA}} \bullet \frac{\text{AG}}{\text{GD}} \bullet \frac{\text{DB}}{\text{BC}} = -1$$ ***M1***

$$\frac{\text{CE}}{\text{EA}} \bullet \frac{2}{1} \bullet \frac{1}{3} = 1$$ ***A1***

$$\frac{\text{CE}}{\text{EA}} = \frac{3}{2}$$ ***AG***

[2 marks]

continued ...

Question 2 continued

(b) (i) using Ceva's theorem in ΔABC,

$\frac{CE}{EA} \bullet \frac{AF}{FB} \bullet \frac{BD}{DC} = 1$ ***M1***

$\frac{3}{2} \bullet \frac{AF}{FB} \bullet \frac{1}{2} = 1$ ***A1***

$\frac{AF}{FB} = \frac{4}{3}$ ***A1***

(ii) using Menelaus' theorem in ΔABE, with traversal (FC), ***M1***

$\frac{AF}{FB} \bullet \frac{BG}{GE} \bullet \frac{EC}{CA} = -1$ ***A1***

$\frac{4}{3} \bullet \frac{BG}{GE} \bullet \frac{3}{5} = 1$ ***A1***

$\frac{BG}{GE} = \frac{5}{4}$ ***A1***

[7 marks]

Sub-total [9 marks]

Part B

using Ptolemy's theorem in PAEC, ***M1***
$PA \bullet EC + AE \bullet PC = PE \bullet AC$ ***A1***
since $EC = AE = AC$, ***M1***
$PE = PA + PC$ ***A1***
similarly for PBDF, ***M1***
$PB \bullet DF + BD \bullet PF = PD \bullet BE$ ***(A1)***
$PD = PB + PF$ ***A1***
adding these results,
$PE + PD = PA + PB + PC + PF$ ***AG***

[7 marks]

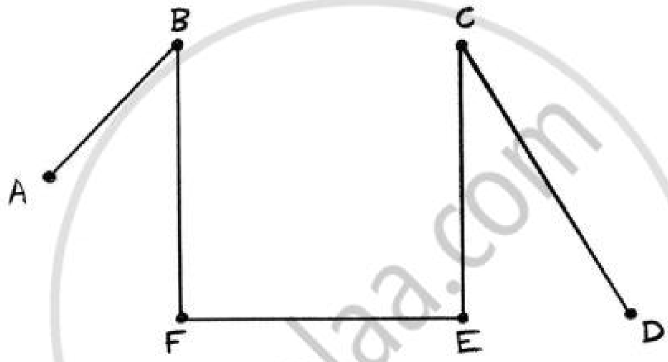
Total [16 marks]

3. (a) (i) *G* has an Eulerian trail because it has 2 vertices of odd degree and the remaining vertices of even degree ***R1***
G does not have an Eulerian circuit because not all vertices are of even degree ***R1***

(ii) BAFBCFECDE ***A1***

[3 marks]

(b) (i) the edges are added in the order
FE ***A1***
CE
AB ***A1***
BF, CD
or CD, BF ***A1***



A1

(ii) minimum weight is 19 ***A1***

[5 marks]

(c)

Step	Vertices labelled	Working values	
1	A	A(0), B – 4, F – 7	***M1A1***
2	AB	A(0), B(4), C – 12, F – 7	***A1***
3	ABF	A(0), B(4), F(7), C – 11, E – 9	***A1***
4	ABFE	A(0), B(4), F(7), E(9), C – 11, D – 18	***A1***
5	ABFEC	A(0), B(4), F(7), E(9), C(11), D – 16	***A1***
6	ABFECD	A(0), B(4), F(7), E(9), C(11), D(16)	***A1***

the shortest path is AFCD ***A2***
the weight is 16 ***A1***

[10 marks]

Total [18 marks]

4. (a) (i) $\bar{x} = \frac{1071}{21} = 51$ *A1*

$S_{n-1}^2 = \frac{54705}{20} - \frac{1071^2}{20 \times 21} = 4.2$ *M1A1*

(ii) degrees of freedom $= 20$; t-value $= 2.086$ *(A1)(A1)*

95 % confidence limits are

$51 \pm 2.086\sqrt{\frac{4.2}{21}}$ *(M1)(A1)*

leading to [50.1, 51.9] *A1*

[8 marks]

(b) $\text{Var}(S_{n-1}) > 0$ *A1*

$\text{E}(S_{n-1}^2) = \sigma^2$ *(A1)*

substituting in the given equation,

$\sigma^2 - \text{E}(S_{n-1})^{\ 2} > 0$ *M1*

it follows that

$\text{E}(S_{n-1}) < \sigma$ *A1*

this shows that S_{n-1} is not an unbiased estimator for σ since that would require $=$ instead of $<$ *R1*

[5 marks]

Total [13 marks]

5. (a) (i) mean $= 6$ ***(A1)***

$\mathrm{P}(5 \text{ customers before } 10{:}00) = \frac{6^5}{5!}\mathrm{e}^{-6} = 0.161$ ***(M1)A1***

(ii) $\mathrm{P}(2 \text{ in } 30 \text{ mins} \mid 5 \text{ in } 60 \text{ mins}) = \frac{\mathrm{P}(2 \text{ in } 30 \text{ mins}) \times \mathrm{P}(3 \text{ in next } 30 \text{ mins})}{\mathrm{P}(5 \text{ in } 60 \text{ mins})}$ ***(M1)***

$$= \frac{\frac{3^2}{2!}\mathrm{e}^{-3} \times \frac{3^3}{3!}\mathrm{e}^{-3}}{\frac{6^5}{5!}\mathrm{e}^{-6}}$$ ***A1A1***

$= \frac{5}{16}$ (accept 0.312 or 0.313) ***A1***

[7 marks]

(b) (i) $\mathrm{P}(T > t) = \mathrm{P}\ \ 0 \text{ or } 1 \text{ arrivals in } [0, t]\ = \left(1 + \frac{t}{10}\right)\mathrm{e}^{-\frac{t}{10}}$ ***R1***

(ii) the distribution function is given by

$F(t) = 1 - \left(1 + \frac{t}{10}\right)\mathrm{e}^{-\frac{t}{10}}$ ***M1A1***

the probability density is given, for $t > 0$, by

$f(t) = F'(t)$ ***(M1)***

$= \frac{1}{10}\left(1 + \frac{t}{10}\right)\mathrm{e}^{-\frac{t}{10}} - \frac{1}{10}\mathrm{e}^{-\frac{t}{10}}$ ***A1***

$= \frac{t}{100}\mathrm{e}^{-\frac{t}{10}}$ ***A1***

(iii) $\mathrm{E}(T) = \frac{1}{100}\int_0^\infty t^2 \mathrm{e}^{-\frac{t}{10}}\,\mathrm{d}t$ ***M1***

$= -\frac{1}{10}\left[t^2 \mathrm{e}^{-\frac{t}{10}}\right]_0^\infty + \frac{1}{10}\int_0^\infty 2t\mathrm{e}^{-\frac{t}{10}}\,\mathrm{d}t$ ***M1A1***

$= -2\left[t\mathrm{e}^{-\frac{t}{10}}\right]_0^\infty + 2\int_0^\infty \mathrm{e}^{-\frac{t}{10}}\,\mathrm{d}t$ ***A1A1***

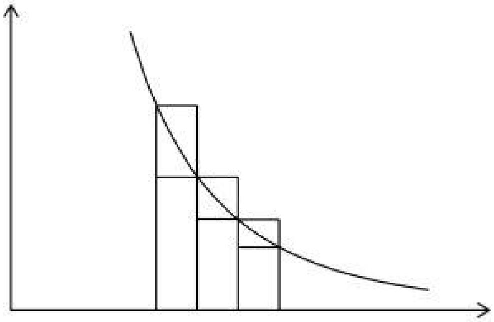
$= -20\left[\mathrm{e}^{-\frac{t}{10}}\right]_0^\infty = 20$ ***A1***

Note: Accept a method based on adding two exponential variables.

[12 marks]

Total [19 marks]

6. (a)



(M1)

total area of 'upper' rectangles

$= \frac{1}{n^4} \times 1 + \frac{1}{(n+1)^4} \times 1 + \frac{1}{(n+2)^4} \times 1 + \ldots = \sum_{r=n}^{\infty} \frac{1}{r^4}$ ***M1A1***

total area of 'lower' rectangles

$= \frac{1}{(n+1)^4} \times 1 + \frac{1}{(n+2)^4} \times 1 + \frac{1}{(n+3)^4} \times 1 + \ldots = \sum_{r=n+1}^{\infty} \frac{1}{r^4}$ ***A1***

the total area under the curve from $x = n$ to infinity lies between these two sums hence $\sum_{r=n+1}^{\infty} \frac{1}{r^4} < \int_n^{\infty} \frac{\mathrm{d}x}{x^4} < \sum_{r=n}^{\infty} \frac{1}{r^4}$ ***R1AG***

[5 marks]

(b) first evaluate the integral

$\int_n^{\infty} \frac{\mathrm{d}x}{x^4} = -\left[\frac{1}{3x^3}\right]_n^{\infty} = \frac{1}{3n^3}$ ***M1A1***

it follows that

$\sum_{r=n+1}^{\infty} \frac{1}{r^4} < \frac{1}{3n^3}$ ***A1***

adding $\sum_{r=1}^{n} \frac{1}{r^4}$ to both sides, ***M1***

$S < \sum_{r=1}^{n} \frac{1}{r^4} + \frac{1}{3n^3}$ ***A1***

similarly,

$\sum_{r=n}^{\infty} \frac{1}{r^4} > \frac{1}{3n^3}$ ***A1***

adding $\sum_{r=1}^{n-1} \frac{1}{r^4}$ to both sides, ***M1***

$S > \sum_{r=1}^{n-1} \frac{1}{r^4} + \frac{1}{3n^3}$ ***A1***

hence the value of S lies between

$\sum_{r=1}^{n-1} \frac{1}{r^4} + \frac{1}{3n^3}$ and $\sum_{r=1}^{n} \frac{1}{r^4} + \frac{1}{3n^3}$ ***AG***

[8 marks]

continued …

Question 6 continued

(c) (i) putting $n = 8$, we find that
$S < 1.08243\ldots$ and $S > 1.08219\ldots$ ***A1A1***
it follows that $S = 1.082$ to 3 decimal places ***A1***

(ii) substituting this value of *S*,
$N \approx \frac{\pi^4}{1.082} \approx 90.0268\ldots$ ***M1A1***
$N = 90$ ***A1***

[6 marks]

(d) **EITHER**

successive partial sums are
1 ***M1***
0.9375
0.9498…
0.9459…
0.9475…
0.9467…
0.9471… ***A1***
it follows that $T = 0.947$ correct to 3 decimal places ***A1***

OR

$T = S - \frac{2}{16}S$ ***M1A1***
$= 0.9471\ldots$ using part (c)(i) or $0.94703\ldots$ using the sum given in part (c)(ii)
it follows that $T = 0.947$ correct to 3 decimal places ***A1***

[3 marks]

Total [22 marks]