GS-2010 (Computer & Systems Sciences)

Full Name : _____

Reference Code : _____

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **COMPUTER & SYSTEMS SCIENCES**
December 13, 2009

Duration : Three hours (3 hours)

Please read all instructions carefully before you attempt the questions.

1. Write your FULL NAME and REFERENCE CODE in block letters on this page and also fill-in all details on the ANSWER SHEET.

2. The Answer Sheet is machine-readable. Please read the instructions on the reverse of the answer sheet before you start filling it up. Only use HB pencils to fill-in the answer sheet.

3. This question paper consists of three (3) parts. **Part-A** contains twenty (20) questions and **must be attempted** by all candidates. **Part-B & Part-C** contain twenty (20) questions each, directed towards candidates for Computer Science and Systems Science, respectively. **ATTEMPT EITHER PART-B OR PART-C BUT NOT BOTH.**

4. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. All questions carry equal marks. A correct answer for a question will give you +4 marks, a wrong answer will give you -1 mark, and a question not answered will not get you any marks. Do not mark more than one circle for any question : this will be treated as a wrong answer.

5. We advise you to first mark the correct answers in the QUESTION SHEET and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

6. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an invigilator. Please do not scribble or do rough work on the reverse of your hall ticket. If found, the hall ticket will be retained.

7. **Use of calculators is NOT permitted.**

8. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

9. **This set of question paper must be returned along with your answer sheet and all extra rough sheets used.**

Part A
Common Questions

1. A box contains 731 black balls and 2000 white balls. The following process is to be repeated as long as possible. Arbitrarily select two balls from the box. If they are of the same colour, throw them out and put a black ball into the box (enough extra black balls are available to do this). If they are of different colours, place the white ball back into the box and throw the black ball away. Which of the following is correct ?

   (a) The process can be applied indefinitely without any a priori bound

   (b) The process will stop with a single white ball in the box

   (c) The process will stop with a single black ball in the box

   (d) The process will stop with the box empty

   (e) None of the above

2. The hour hand and the minute hands of a clock meet at noon and again at midnight. In between they meet N times, where N is:

   (a) 6

   (b) 11

   (c) 12

   (d) 13

   (e) None of the above.

3. The function $f(x) = 2.5 log_e \left(2 + exp(x^2 - 4x + 5)\right)$ attains a minimum at $x =$

   (a) 0

   (b) 1

   (c) 2

   (d) 3

   (e) 4

4. If the bank receipt is forged, then Mr. M is liable. If Mr. M is liable, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. The bank will loan him money.

   Which of the following can be concluded from the above statements ?

   (a) Mr. M is liable

   (b) The receipt is not forged

   (c) Mr. M will go bankrupt

   (d) The bank will go bankrupt

   (e) None of the above.

5. A is a symmetric positive definite matrix (i.e., $x^T A x > 0$ for all nonzero x). Which of the following statements is false?

   (a) At least one element is positive.

   (b) All eigenvalues are positive real.

   (c) Sum of the diagonal elements is positive.

   (d) $det(A)$ is positive.

   (e) None of the above.

6. Given 10 tosses of a coin with probability of head $= .4 = (1 -$ the probability of tail), the probability of at least one head is

   (a) $(.4)^{10}$

   (b) $1 - (.4)^{10}$

   (c) $1 - (.6)^{10}$

   (d) $(.6)^{10}$

   (e) $10(.4)(.6)^9$

7. The limit of $10^n/n!$ as $n \to \infty$ is

   (a) 0

   (b) 1

   (c) e

   (d) 10

   (e) ∞

8. Which of the following is NOT necessarily true? (Notation: The symbol "$\neg$" denotes negation; $P(x, y)$ means that for given x and y, the property $P(x, y)$ is true.)

   (a) $(\forall x \forall y P(x, y)) \Rightarrow (\forall y \forall x P(x, y))$

   (b) $(\forall x \exists y \neg P(x, y)) \Rightarrow \neg(\exists x \forall y P(x, y))$

   (c) $(\exists x \exists y P(x, y)) \Rightarrow (\exists y \exists x P(x, y))$

   (d) $(\exists x \forall y P(x, y)) \Rightarrow (\forall y \exists x P(x, y))$

   (e) $(\forall x \exists y P(x, y)) \Rightarrow (\exists y \forall x P(x, y))$

9. A table contains 287 entries. When any one of the entries is requested, it is encoded into a binary string and transmitted. The number of bits required is

   (a) 8

   (b) 9

   (c) 10

   (d) Cannot be determined from the given information.

   (e) None of the above.

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10. A drawer contains 2 blue, 4 red and 2 yellow balls. No two balls have the same radius. If two balls are randomly selected from the drawer, what is the probability that they will be of the same colour?

    (a) 2/7
    (b) 2/5
    (c) 3/7
    (d) 1/2
    (e) 3/5

11. The length of a vector $\mathbf{x} = (x_1, \ldots, x_n)$ is defined as

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2} \quad .$$

Given two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, which of the following measures of discrepancy between $\mathbf{x}$ and $\mathbf{y}$ is insensitive to the length of the vectors ?

    (a) $\|\mathbf{x} - \mathbf{y}\|$
    (b) $\|\mathbf{x} - \mathbf{y}\|/\|\mathbf{x}\| \, \|\mathbf{y}\|$
    (c) $\|\mathbf{x}\| - \|\mathbf{y}\|$
    (d) $\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\|$
    (e) None of the above

12. The coefficient of x^3 in the expansion of $(1 + x)^3(2 + x^2)^{10}$ is

    (a) 2^{14}
    (b) 31
    (c) $\binom{3}{3} + \binom{10}{1}$
    (d) $\binom{3}{3} + 2\binom{10}{1}$
    (e) $\binom{3}{3}\binom{10}{1}2^9$

13. A cube whose faces are colored is split into 1000 small cubes of equal size. The cubes thus obtained are mixed thoroughly. The probability that a cube drawn at random will have *exactly two* colored faces is:

    (a) 0.096
    (b) 0.12
    (c) 0.104
    (d) 0.24
    (e) none of the above

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14. A marine biologist wanted to estimate the number of fish in a large lake. He threw a net and found 30 fish in the net. He marked all these fish and released them into the lake. The next morning he again threw the net and this time caught 40 fish, of which two were found to be marked. The (approximate) number of fish in the lake is:

    (a) 600

    (b) 1200

    (c) 68

    (d) 800

    (e) 120

15. Let A, B be sets. Let $\overline{A}$ denote the complement of set A (with respect to some fixed universe), and $(A - B)$ denote the set of elements in A which are not in B. Set $(A - (A - B))$ is equal to:

    (a) B

    (b) $A \cap \overline{B}$

    (c) $A - B$

    (d) $A \cap B$

    (e) $\overline{B}$

16. Let the characteristic equation of a matrix M be $\lambda^2 - \lambda - 1 = 0$. Then

    (a) M^{-1} does not exist.

    (b) M^{-1} exists but cannot be determined from the data

    (c) $M^{-1} = M + I$

    (d) $M^{-1} = M - I$

    (e) M^{-1} exists and can be determined from the data but the choices (c) and (d) are incorrect.

17. Suppose there is a sphere with diameter at least 6 inches. Through this sphere we drill a hole along a diameter. The part of the sphere lost in the process of drilling the hole looks like two caps joined to a cylinder, where the cylindrical part has length 6 inches. *It turns out that the volume of the remaining portion of the sphere does not depend on the diameter of the sphere.* Using this fact, determine the volume of the remaining part.

    (a) 24π cu. inches

    (b) 36π cu. inches

    (c) 27π cu. inches

    (d) 32π cu. inches

    (e) 35π cu. inches

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18. Let X be a set of size n. How many pairs of sets (A, B) are there that satisfy the condition $A \subseteq B \subseteq X$?

(a) 2^{n+1}

(b) 2^{2n}

(c) 3^n

(d) $2^n + 1$

(e) 3^{n+1}

19. Karan tells truth with probability 1/3 and lies with probability 2/3. Independently, Arjun tells truth with probability 3/4 and lies with probability 1/4. Both watch a cricket match. Arjun tells you that India won, Karan tells you that India lost. What probability will you assign to India's win?

(a) 1/2

(b) 2/3

(c) 3/4

(d) 5/6

(e) 6/7

20. How many integers from 1 to 1000 are divisible by 30 but not by 16?

(a) 29

(b) 31

(c) 32

(d) 33

(e) 25

Part B
Computer Science

21. For $x \in \{0,1\}$, let $\neg x$ denote the negation of x; that is $\neg x = 1$ if $x = 0$ and $\neg x = 0$ if $x = 1$. If $x \in \{0,1\}^n$, then $\neg x$ denotes the component-wise negation of x; that is

$$(\neg x)_i = \neg x_i, \quad \text{for } i = 1, 2, \ldots, n.$$

Consider a circuit C computing a function $f : \{0,1\}^n \rightarrow \{0,1\}$ using AND, OR and NOT gates. Let D be the circuit obtained from C by replacing each AND gate by an OR gate and replacing each OR gate by an AND. Suppose D computes the function g. Which of the following is true for all inputs x?

   (a) $g(x) = \neg f(x)$

   (b) $g(x) = f(x)$ AND $f(\neg x)$

   (c) $g(x) = f(x)$ OR $f(\neg x)$

   (d) $g(x) = \neg f(\neg x)$

   (e) None of the above

22. Let L consist of all binary strings beginning with a 1 such that its value when converted to decimal is divisible by 5. Which of the following is true?

   (a) L can be recognised by a deterministic finite state automaton.

   (b) L can be recognised by a non-deterministic finite state automaton but not by a deterministic finite state automaton.

   (c) L can be recognised by a deterministic push-down automaton but not by a non-deterministic finite state automaton.

   (d) L can be recognised by a non-deterministic push-down automaton but not by a deterministic push-down automaton.

   (e) L cannot be recognised by any push down automaton.

23. Suppose you are given n numbers and you sort them in descending order as follows: First find the maximum. Remove this element from the list and find the maximum of the remaining elements, remove this element, and so on, until all elements are exhausted. How many comparisons does this method require in the worst case?

   (a) linear in n.

   (b) $O(n^2)$ but not better.

   (c) $O(n \log n)$.

   (d) Same as heap sort.

   (e) $O(n^{1.5})$ but not better.

24. Consider the following program operating on four variables, u, v, x and y, and two constant X and Y.

$x, y, u, v := X, Y, Y, X;$
while$(x \neq y)$
do

    if $(x > y)$ then $x, v := x - y, v + u$
    else if $(y > x)$ then $y, u := y - x, u + v;$

od;
print $((x + y)/2);$ print $((u + v)/2)$

Given $X > 0 \wedge Y > 0$, pick the true statement out of the following:

(a) the program prints the gcd(X, Y) and the first prime larger than both X and Y

(b) the program prints gcd(X, Y) followed by lcm(X, Y)

(c) the program prints gcd(X, Y) followed by $(1/2) \times$ lcm(X, Y)

(d) the program prints $(1/2) \times$ gcd(X, Y) followed by $(1/2) \times$ lcm(X, Y)

(e) the program does none of the above

25. Which of the following problems is decidable ? (Here, CFG means context free grammar and CFL means context free language.)

(a) Given a CFG G, find whether L(G) = R where R is regular set

(b) Given a CFG G, find whether L(G) = {}

(c) Find whether the intersection of two CFLs is empty

(d) Find whether the complement of CFL is a CFL

(e) Find whether CFG G1 and CFG G2 generate the same language, i.e. L(G1) = L(G2)

26. Suppose there is a balanced binary search tree with n nodes, where at each node, in addition to the key, we store the number of elements in the subtree rooted at that node. Now, given two elements a and b, such that $a < b$, we want to find the number of elements x in the tree that lie between a and b, that is, $a \leq x \leq b$. This can be done with (choose the best solution)

(a) $O(\log n)$ comparisons and $O(\log n)$ additions.

(b) $O(\log n)$ comparisons but no furhter additions.

(c) $O(\sqrt{n})$ comparisons but $O(\log n)$ additions.

(d) $O(\log n)$ comparisons but a constant number of additions.

(e) $O(n)$ comparisons and $O(n)$ additions, using depth-first-search.

27. Consider the Insertion Sort procedure given below, which sorts an array L of size n (≥ 2) in ascending order.

begin
    for $xindex := 2$ **to** n **do**
        $x := L[xindex]$;
        $j := xindex - 1$;
        while $j > 0$ **and** $L[j] > x$ **do**
            $L[j+1] := L[j]$;
            $j := j - 1$
        end {*while*}
        $L[j+1] := x$;
    end {*for*}
end

It is known that Insertion Sort makes *at most* $n(n-1)/2$ comparisons. Which of the following is true?

(a) There is no input on which Insertion Sort makes $n(n-1)/2$ comparisons.

(b) Insertion Sort makes $n(n-1)/2$ comparisons when the input is already sorted in ascending order.

(c) Insertion Sort makes $n(n-1)/2$ comparisons only when the input is sorted in descending order.

(d) There are more than one input orderings where Insertion Sort makes $n(n-1)/2$ comparisons.

(e) Insertion sort makes $n(n-1)/2$ comparisons whenever all the elements of L are not distinct.

28. Consider the concurrent program

```
x:=1;
   cobegin
      x:= x + 3 || x := x + x + 2
   coend
```

Reading and writing of variables is atomic, but the evaluation of an expression is **not** atomic. The set of possible values of variable x at the end of the execution of the program is:

(a) $\{4\}$

(b) $\{8\}$

(c) $\{4, 7, 10\}$

(d) $\{4, 7, 8, 10\}$

(e) $\{4, 7, 8\}$

29. Suppose you are given an array A with $2n$ numbers. The numbers in odd positions are sorted in ascending order, that is, $A[1] \le A[3] \le \cdots \le A[2n-1]$; the numbers in even positions are sorted in descending order, that is, $A[2] \ge A[4] \ge \cdots \ge A[2n]$. What is the method you would recommend for determining if a given number is in the array?

(a) Sort the array using quick-sort and then use binary search.

(b) Merge the sorted lists and perform binary search.

(c) Perform a single binary search on the entire array.

(d) Perform separate binary searches on the odd positions and the even positions.

(e) Search sequentially from the end of the array.

30. Consider the following program for summing the entries of the array b : array [0..N-1] of integers, where N is a positive integer. (The symbol '<>' denotes 'not equal to').

```
var
      i,s : integer;

      Program
            i := 0;
            s := 0;
[*]     while i <> N do
            s := s + b[i];
            i := i+1;
            od
```

Which of the following gives the invariant that holds at the beginning of each loop, that is, each time the program arrives at point [*]?

(a) $s = \sum_{j=0}^{N} b[j]$ & $0 \le i \le N$

(b) $s = \sum_{j=0}^{i-1} b[j]$ & $0 \le i < N$

(c) $s = \sum_{j=0}^{i} b[j]$ & $0 < i \le N$

(d) $s = \sum_{j=1}^{N} b[j]$ & $0 \le i < N$

(e) $s = \sum_{j=0}^{i-1} b[j]$ & $0 \le i \le N$

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31. Consider the following computation rules. **Parallel - outermost rule:** Replace all the outermost occurrences of F (i.e., all occurrences of F which do not occur as arguments of other Fs) simultaneously. **Parallel - innermost rule:** Replace all the innermost occurrences of F (i.e., all occurrences of F with all arguments free of F's) simultaneously. Now consider the evaluations of the recursive program over the integers.

```
F(x,y) <==  if x = 0 then 0 else
              [ F(x+1, F(x,y))  *  F(x-1, F(x,y)) ]
```

where the multiplication functions * is extended as follows:

```
0 * w  & w * 0  are 0
a * w  & w * a  are w  (for any non zero integer a)
w * w is w
```

We say that F(x,y)=w when the evaluation of F(x,y) does not terminate. Computing $F(1,0)$ using the parallel-innermost and parallel-outermost rules yields

(a) w and 0 respectively

(b) 0 and 0 respectively

(c) w and w respectively

(d) w and 1 respectively

(e) none of the above

32. Consider the following solution (expressed in Dijkstra's guarded command notation) to the mutual exclusion problem.

```
process P1 is
begin
  loop
    Non_critical_section;
    while not(Turn=1) do skip od;
    Critical_section_1;
    Turn := 2
  end loop
end

||

process P2 is
begin
  loop
    Non_critical_section;
    while not(Turn=2) do skip od;
    Critical_section_2;
    Turn := 1
  end loop
end
```

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Initially, `Turn=1`. Assume that the two processes run forever and that no process stays in its critical or non-critical section infinitely. A mutual exclusion program is correct if it satisfies the following requirements.

(1) Only one process can be in a critical region at a time.

(2) Program is deadlock-free, i.e. if both processes are trying to enter the critical region then at-least one of them does enter the critical region.

(3) Program is starvation-free; i.e. a process trying to enter the critical region eventually manages to do so.

The above mutual exclusion solution:

(a) Does not satisfy the requirement (1).

(b) Satisfies the requirement (1) but does not satisfy the requirement (2).

(c) Satisfies the requirements (1) and (2), but does not satisfy the requirement(3).

(d) Satisfies the requirements (1) and (3), but does not satisfy the requirement (2)

(e) Satisfies all the requirements (1), (2) and (3)

33. In a relational database there are three relations:

- *Customers* = C(CName),
- *Shops* = S(SName),
- *Buys* = B(CName, SName).

Then the Relational Algebra expression (Π is the projection operator)

$$C - \Pi_{\text{CName}}((C \times S) - B)$$

returns the names of

(a) Customers who buy from at least one shop

(b) Customers who buy from at least two shops

(c) Customers who buy from all shops

(d) Customers who do not buy anything at all

(e) None of the above

34. Let r,s,t be regular expressions. Which of the following identities is correct ?

(a) $(r + s)^* = r^* s^*$

(b) $r(s + t) = rs + t$

(c) $(r + s)^* = r^* + s^*$

(d) $(rs + r)^* r = r(sr + r)^*$

(e) $(r^* s)^* = (rs)^*$

35. Consider the following langugages over the alphabet $\{0, 1\}$.

$$L1 = \{x.x^R \mid x \in \{0, 1\}^*\}$$

$$L2 = \{x.x \mid x \in \{0, 1\}^*\}$$

where x^R is the reverse of strng x; e.g. $011^R = 110$. Which of the following is true?

(a) Both $L1$ and $L2$ are regular.

(b) $L1$ is context-free but not regular where as $L2$ is regular.

(c) Both $L1$ and $L2$ are context free and neither is regular.

(d) $L1$ is context free but $L2$ is not context-free.

(e) Both $L1$ and $L2$ are not context-free.

36. In a directed graph, every vertex has exactly seven edges coming in. What can one always say about the number of edges going out of its vertices?

(a) Exactly seven edges leave *every* vertex.

(b) Exactly seven edges leave *some* vertex.

(c) Some vertex has at least seven edges leaving it.

(d) The number of edges coming out of every vertex is odd.

(e) None of the above.

37. Consider the program where a, b are integers with $b > 0$.

```
x:=a; y:=b; z:=0;
  while y > 0 do
    if odd(x) then z:= z+x; y:= y-1
              else y:= y % 2; x:= 2*x
    fi
```

Invariant of the loop is a condition which is true before and after **every** iteration of the loop. In the above program the loop invariant is given by

$$0 \leq y \quad \text{and} \quad z + x * y = a * b$$

Which of the following is true of the program?

(a) The program will not terminate for some values of a, b

(b) The program will terminate with $z = 2^b$

(c) The program will terminate with $z = a * b$

(d) The program will not terminate for some values of a, b but when it does terminate, the condition $z = a * b$ will hold

(e) The program will terminate with $z = a^b$

38. Suppose three coins are lying on a table, two of them with *heads* facing up and one with *tails* facing up. One coin is chosen at random and flipped. What is the probability that after the flip the majority of the coins (i.e. at least two of them) will have *heads* facing up?

   (a) $\frac{1}{3}$

   (b) $\frac{1}{8}$

   (c) $\frac{1}{4}$

   (d) $\frac{1}{4} + \frac{1}{8}$

   (e) $\frac{2}{3}$

39. Suppose a language L is NP-complete. Then which of the following is FALSE.

   (a) $L \in NP$.

   (b) Every problem in P is polynomial time reducible to L.

   (c) Every problem in NP is polynomial time reducible to L.

   (d) The Hamilton cycle problem is polynomial time reducible to L.

   (e) $P \neq NP$ and $L \in P$.

40. Which of the following statements is FALSE ?

   (a) All recursive sets are recursively enumerable

   (b) The complement of every recursively enumerable sets is recursively enumerable

   (c) Every non-empty recursively enumerable set is the range of some totally recursive function

   (d) All finite sets are recursive.

   (e) The complement of every recursive set is recursive

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Part C
Systems Science

41. A linear system could be a composition of

    (a) Two non-linear systems

    (b) a non-causal non-linear system and a linear system

    (c) a time varying non-linear system and a time varying linear system

    (d) All of the above

    (e) None of the above

42. For $x \in [0, \pi/2]$, α for which $\sin(x) \geq x - \alpha x^3$ is

    (a) $\alpha > 1/(2\pi)$

    (b) $\alpha \geq 1/6$

    (c) $\alpha \leq 1/(2\pi)$

    (d) $\alpha = 1/4$

    (e) None of the above

43. Consider two independent random variables X and Y having probability density functions uniform in the interval $[0, 1]$. When $\alpha \geq 1$, the probability that $\max(X, Y) > \alpha \min(X, Y)$ is

    (a) $1/(2\alpha)$

    (b) $\exp(1 - \alpha)$

    (c) $1/\alpha$

    (d) $1/\alpha^2$

    (e) $1/\alpha^3$

44. Let $Y_n = s_n + W_n$ where $\{s_n\}$ is the desired signal bandlimited to $[-W, W]$ and $\{W_n\}$ is a noise component, which is sparse (that is, only few samples are non-zero), bursty (that is, runs of non-zero samples are rare), and its amplitude is large compared to the desired signal. Which of the following filtering techniques is preferable?

    (a) Low pass filter with cutoff at W

    (b) High pass filter with cutoff at W is used first (to estimate W_n) and the output of the high pass filter is subtracted from the input

    (c) Bandpass filter with suitable cutoffs

    (d) The output at time n is chosen to be the median of $\{Y_{n+k}\}_{k=-K}^{K}$ for suitably chosen K

    (e) Both a) and b) are better than the other options

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45. Let $Y(t) = \sum_{n=-\infty}^{\infty} x_n h(t - nT)$. We sample $Y(t)$ at time instants $nT/2$ and let $Y_n = Y(nT/2)$. Which of the following is true?

    (a) $\{Y_n\}$ can be interpreted as the output of a discrete time, linear, time-invariant system with input $\{X_n\}$.

    (b) $\{Y_{2n}\}$ can be interpreted as the output of a discrete time, linear, time-invariant system with input $\{X_n\}$.

    (c) $\{Y_{2n+1}\}$ can be interpreted as the output of a discrete time, linear, time-invariant system with input $\{X_n\}$.

    (d) Both a) and b) above

    (e) Both b) and c) above

46. If we convolve $\sin(t)/t$ with itself, then we get

    (a) $C \sin(t)/t$ for some constant C

    (b) $C \cos(t)/t$ for some constant C

    (c) $C \cos(t)/t^2$ for some constant C

    (d) $C_1 \sin(t)/t^2 + C_2 \cos(t)/t^2$ for some constants C_1, C_2

    (e) None of the above

47. A voltage source with internal resistance R is connected to an inductor L and a capacitor C connected in parallel. The output is the common voltage across the inductor and the capacitor. What is the nature of the transfer function of this system?

    (a) Low pass.

    (b) High pass.

    (c) Band pass.

    (d) Either (a) or (b) depending upon the values of L and C.

    (e) The circuit is not stable and no transfer function exists.

48. Consider a discrete time channel with binary inputs and binary outputs. Let x_n denote the input bit at time n and y_k denote the output bit at time k. The channel operation is such that to produce the output y_n it drops one of the two inputs x_{2n}, x_{2n+1} and outputs the other. Thus $y_n = x_{2n}$ with probability $1/2$ and $y_n = x_{2n+1}$ with probability $1/2$. Suppose we wish to send M messages using length $2N$ inputs. Let $R = \log_2(M)/2N$. Which of the following is true.

    (a) If $R > 1/2$, then we always make errors

    (b) If $R = 1/2$, then there is a transmission scheme such that we do not make any error

    (c) If $R < 1/2$, then there exists a scheme with zero error

    (d) All of the above

    (e) None of the above

49. The $z-$transform of a sequence $\{x_n\}_{n=-\infty}^{\infty}$ is defined to be $X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$. The z-transform of the sequence $y_n = x_{2n+1}$ is

(a) $Y(z) = z(X(z) - X(-z))/2$

(b) $Y(z) = \sqrt{z}(X(\sqrt{z}) - X(-\sqrt{z}))/2$

(c) $Y(z) = z^2(X(z^2) - X(-z^2))/2$

(d) $Y(z) = z(X(\sqrt{z}) - X(-\sqrt{z}))/2$

(e) $Y(z) = (X(\sqrt{z}) - X(-\sqrt{z}))/2$

50. H is a circulant matrix (row n is obtained by circularly shifting row 1 to the right by n positions) and F is the DFT matrix. Which of the following is true?

(a) FHF^H is circulant, where F^H is the inverse DFT matrix.

(b) FHF^H is tridiagonal

(c) FHF^H is diagonal

(d) FHF^H has real entries

(e) None of the above

51. Consider

$$F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 2.1 \\ 1.2 \\ 1.3 \\ 2.4 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$

The inner product between Fx and Fy is

(a) 0

(b) 1

(c) -1

(d) -1.2

(e) None of the above

52. Consider a system with input $x(t)$ and the output $y(t)$ is given by

$$y(t) = x(t) - \sin(t)x(t-1) - 0.5x(t+2) + 1.$$

The system is

(a) Non-linear

(b) Non-causal

(c) Time varying

(d) All of the above

(e) None of the above

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53. Output of a linear system with input $x(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} h(t, \tau) x(\tau).$$

The system is time invariant if

(a) $h(t, \tau) = h(t - \tau)$

(b) $h(t, \tau) = h(\tau)$

(c) $h(t, \tau) = h(t)$

(d) $h(t, \tau) = \text{constant}$

(e) $h(t, \tau)$ is a continuous function of t

54. Define $sign(x) = 0$ for $x = 0$, $sign(x) = 1$ for $x > 0$ and $sign(x) = -1$ for $x < 0$. For $n \geq 0$, let
$$Y_n = sign(X(n) - Z_n),$$
where $Z_n = \sum_{k \leq n} Y_k$, $Z_0 = 0$, $X(t) = t$ for $t < 5.5$ and $X(t) = 5.5$ for $t \geq 5.5$. Then the sequence Y_n is equal to

(a) 0, followed by six 1's, followed by -1,1,-1,1,....

(b) 0, followed by five 1's, followed by -1,1,-1,1,....

(c) 0,1,-1,1,-1,...

(d) 0, 1, 1, 1, -1, 1, -1, 1, ...

(e) None of the above

55. Let $\imath = \sqrt{-1}$. Then $\imath^{\imath}$ could be

(a) $\exp(\pi/2)$

(b) $\exp(\pi/4)$

(c) Can't determine

(d) Takes infinite values

(e) Is a complex number

56. Consider two independent random variables X and Y having probability density functions uniform in the interval $[0, 1]$. The probability that $X + Y > 1.5$ is

(a) 1/4

(b) 1/8

(c) 1/3

(d) $\Pr\{X + Y < 0.25\}$

(e) None of the above

57. Let $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. Then the limit

$$\lim_{n \to \infty} \left(\sum_{i=1}^{k} a_i^n \right)^{1/n}$$

is

(a) 0

(b) ∞

(c) a_k

(d) a_1

(e) $(\sum_{i=1}^{k} a_k)/k$

58. Under what conditions is the following inequality true for $a, b > 0$

$$\log_e(a + b) \geq \lambda \log_e(a/\lambda) + (1 - \lambda) \log_e(b/(1 - \lambda))$$

(a) $\lambda = 0.5$

(b) $0 < a/\lambda \leq 1$, $b/(1 - \lambda) > 0$

(c) $a/\lambda > 0$, $0 < b/(1 - \lambda) \leq 1$

(d) All of the above

(e) None of the above

59. Let us define an interval $A(n)$ as a function of n as $A(n) = (-1/n, 1/n)$. Then the set of points that lie in the intersection of A_n's, $n = 1, ..., \infty$

(a) is an interval

(b) is a single point

(c) is an empty set

(d) cannot be determined

(e) has two disjoint intervals

60. The function $f(t)$ is a convolution of t^2 with $\exp(-t^2/2)/\sqrt{2\pi}$. Its derivative is

(a) $2t$

(b) t^2

(c) $2t + te^{-t^2/2}$

(d) Does not have a simple closed form expression

(e) None of the above