Instructions

(1) This question paper consists of two parts: Part A and Part B and carries a total of 100 Marks.

(2) There is no negative marking.

(3) Part A carries 20 multiple choice questions of 2 marks each. Answer all questions in Part A.

(4) Answers to Part A are to be marked in the OMR sheet provided.

(5) For each question, darken the appropriate bubble to indicate your answer.

(6) Use only HB pencils for bubbling answers.

(7) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(8) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

(9) Part B has 24 questions. Answer any 12 in this part. Each question carries 5 marks.

(10) Answers to Part B are to be written in the separate answer book provided.

(11) Candidates are asked to fill in the required fields on the sheet attached to the answer book.

(12) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the set of integers, real numbers, rational numbers and complex numbers respectively.

(13) If G is a group, then $O(G)$ denotes the order of G.

MATHEMATICS

PART A

(1) The ordinary differential equation $g' = 2g$ with $g(0) = a$ has

(A) the solution $g(x) = 2\exp(ax)$,
(B) the solution $g(x) = (\exp(ax) - \exp(-ax))/2$,
(C) the solution $g(x) = a\exp(2x)$,
(D) no solution.

(2) Let $x(t)$ and $y(t)$ be C^∞ functions on $\mathbb{R}$ and let $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Let A be a 2×2 real constant matrix such that $z'(t) = Az(t)$ for all $t \in \mathbb{R}$. Let λ be an eigenvalue of A with corresponding eigenvector v. Then a solution for $z(t)$ is

(A) $\exp(\lambda t)v$,
(B) $\lambda\exp(\lambda t)v$,
(C) $\exp(-\lambda t)v$,
(D) $\exp(i\lambda t)v$.

(3) Let f be a non-constant entire function such that $|f(z)| = 1$ for every z with $|z| = 1$. Then

(A) f has a zero in the open unit disc.
(B) f always has a zero outside the closed unit disc.
(C) f need not have any zero.
(D) any such f has exactly one zero in the open unit disc.

(4) Let f have a pole of order 2 at 0 and let g be an analytic function in a neighbourhood of 0 having a zero of order 3 at 0. Then the function $f(z)g(z)$ has

(A) a pole of order 2 at 0,
(B) a zero of order 2 at 0,
(C) a pole of order 1 at 0
(D) a zero of order 1 at 0.

(5) Let f be an entire function whose values lie in a straight line in the complex plane. Then

(A) f is necessarily identically equal to 0,
(B) f is constant,
(C) f is a Möbius map,
(D) f is a linear function.

(6) Given a non-constant complex valued function $f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$ with u and v being real valued twice continuously differentiable functions, define

$$\partial f = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) \text{ and } \bar{\partial} f = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

Then f is analytic if

(A) $\partial f = 0$,
(B) $\bar{\partial} f = 0$,
(C) $\partial f = \bar{\partial} f$,
(D) $\partial f = -\bar{\partial} f$.

(7) Let $f : [-1,1] \to \mathbb{R}$ be a continuous function such that $\int_{-1}^{x} f(t)dt = 0$ for all $x \in [-1,1]$. Then

(A) f is identically 0,
(B) f is a non-zero odd function,
(C) f is a non-zero even function,
(D) f is a non-zero periodic function.

(8) Let A be a closed infinite subset of $\mathbb{R}^n$. Then

(A) A is always the closure of its interior,
(B) A is always compact,
(C) A is always the closure of a countable set,
(D) A is always a bounded set.

(9) For a continuous function $f : \mathbb{R} \to \mathbb{R}$, let $Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$. Then

(A) $Z(f)$ is always a compact set,
(B) $Z(f)$ is always a closed set,
(C) $Z(f)$ is always a connected set,
(D) $Z(f)$ is always an open set.

(10) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function with $f(0) = f(1) = f'(0) = 0$. Then

(A) f'' has no zeros in $[0.1]$,
(B) $f''(x) = 0$ for some $x \in (0, 1)$,
(C) $f''(0)$ is always 0,
(D) $f''(0)$ is always 1.

(11) If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear map, then the image of the unit square is always

(A) a square,
(B) a rectangle,
(C) a disc,
(D) a parallelogram.

(12) Let A be a 3×3 real matrix such that $A^2 = -I_3$ where I_3 is the 3×3 identity matrix. Such an A

(A) is diagonalizable,
(B) is orthogonal,
(C) does not exist
(D) is symmetric.

(13) Let

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ where } 0 \neq a \in \mathbb{R}.$$

If B is another 2×2 real matrix which commutes with A, then the eigenvalues of B are always

(A) equal,
(B) distinct,
(C) equal to 0,

(D) equal to 1.

(14) If A is a real $n \times n$ matrix satisfying $A^3 = A$, then Trace of A is always

(A) n,

(B) 0,

(C) $-n$,

(D) an integer in the set $\{-n, -(n-1), \ldots, -1, 0, 1, \ldots, n\}$.

(15) Let (A, B) be a pair of $n \times n$ matrices such that $AB - BA = I_n$ where I_n is the $n \times n$ identity matrix.

(A) Then A and B are simultaneously diagonalizable,

(B) Such a pair (A, B) does not exist,

(C) Rank A = Rank B,

(D) $\det AB = 1/2$.

(16) Let G be an abelian group. If a and b are two elements of order 8 and 10 respectively, then the order of the element $a^{-1}b$ is

(A) 80,

(B) 18,

(C) 2,

(D) 40.

(17) Let $\mathbb{C}^*$ be the group $\mathbb{C} \setminus \{0\}$. Then a finite subgroup of $\mathbb{C}^*$

(A) is contained in $\mathbb{R}^*$,

(B) consists of only -1 and 1,

(C) is contained in $\mathbb{Q}^*$,

(D) is contained in $\{z \in \mathbb{C} : |z| = 1\}$.

(18) If G is a group of order 20, then the number of subgroups of G of order 5 is

(A) 1,

(B) 4,

(C) 5,

(D) 2.

(19) If the order of every non-trivial element in a group is n, then

(A) n is necessarily a prime number,
(B) n can be any odd number,
(C) n is an even number,
(D) n can be any positive integer.

(20) Let G_1 and G_2 be two groups such that $O(G_1)$ and $O(G_2)$ are relatively prime. If $f : G_1 \to G_2$ is a homomorphism, then

(A) f is necessarily trivial,
(B) f is necessarily onto,
(C) f is necessarily injective,
(D) f is an isomorphism.

Part B

(1) Let (X, d_1) and (Y, d_2) be two metric spaces and let $f : X \to Y$ be an onto continuous function satisfying

$$d_1(x, y) \leq d_2(f(x), f(y)) \text{ for all } x, y \in X.$$

Prove that if (X, d_1) is complete then (Y, d_2) is also complete.

(2) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Suppose that there is a $c < \infty$ such that $|f'(x)| \leq c$ for all $x \in (a, b)$. Prove that f extends continuously to $[a, b]$.

(3) Let $\{x_n\}$ be a sequence in a metric space. Prove that the sequence $\{x_n\}$ converges if and only if every proper subsequence of $\{x_n\}$ converges.

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with $f(0) = 0$. Prove that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = xg(x)$ for all x.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a one-to-one differentiable function. Prove that f is strictly increasing or strictly decreasing.

(6) A function $g : \mathbb{R} \to \mathbb{R}$ is said to be a *step function* if there are real numbers a, b and c such that

$$\begin{aligned} g(x) &= c \quad \text{if} \quad b \leq x \leq a \\ &= 0 \quad \text{if} \quad x < b \text{ or } x > a. \end{aligned}$$

Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 f(x)g(x)dx = 0$$

for any step function g. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

(7) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. Prove that f is (complex) analytic as a function from $\mathbb{C}$ to $\mathbb{C}$ if and only if the matrix of $f'(x)$ as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ commutes with the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $x \in \mathbb{R}^2$.

(8) Calculate $\int_0^{2\pi} \exp(\exp(i\theta))\, d\theta$.

(9) Prove that the function $f(z) = \overline{\exp(1/\overline{z})}$ is analytic in $\mathbb{C} \setminus \{0\}$.

(10) Is there an entire function $g(z)$ such that $g(z) = 1/z$ for $|z| \geq 1$?. Justify your answer.

(11) Prove that

$$\int_{|z|=r} \frac{dz}{z^3 + 1}$$

is a constant for large r and find its value.

(12) Let A be a 2×2 real matrix. Suppose that $A^2 = I$, where I is the identity matrix. Prove that A is diagonalizable.

(13) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $< Tx, Ty >= 0$ if $< x, y >= 0$. Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$.

(a) Show that $< T(e_i - e_j), T(e_i + e_j) >= 0$ 1 Mark

(b) Show that $< Te_i, Te_i >$ is a constant independent of i. 2 Marks

(c) Let $k =< Te_1, Te_1 >$. Show that $< Tx, Ty >= k < x, y >$ for all $x,\ y \in \mathbb{R}^n$ 2 Marks

(14) Suppose that T is a real $n \times n$ matrix such that $T^m x = 0$ and $T^{m-1} x \neq 0$ for some vector x. Prove that $x, Tx, .., T^{m-1}x$ are linearly independent.

(15) Suppose that A is a real $n \times n$ matrix such that Trace $(A^t B) = 0$ for any real $n \times n$ matrix B. Prove that $A = 0$.

(16) Let A be an $n \times n$ real matrix such that $A^2 = I$. Prove that Rank $(I + A)$ + Rank $(I - A) = n$.

(Hint: By a result of Frobenius, if A and B are two $n \times n$ matrices then Rank $(AB) \geq$ Rank A + Rank $B - n$).

(17) Let G be a finite group and H a subgroup such that G/H has only two elements. Prove that H is a normal subgroup of G.

(18) Let R be a finite ring and a an element of R which is not a zero divisor. Prove that a is invertible.

(19) Let G be a group and suppose that $g \in G$ is the *unique* element of order 2. Prove that g belongs to the center of G, i.e., g commutes with every element of G.

(20) Let G be a finite abelian group of odd order. Define the map $f : G \to G$ by $f(g) = g^2$. Prove that f is an automorphism.

(21) Let F be a finite set. Let $\mathcal{A}$ consist of all functions from F to the complex plane. Prove that $\mathcal{A}$ is a ring and find all the invertible elements.

(22) Consider the functions $f(x) = x^3$ and $g(x) = x^2|x|$ defined on the interval $[-1, 1]$.

(a) Show that their Wronskian $W(f, g)$ vanishes identically. 3 Marks

(b) Show that f and g are not linearly dependent. 2 Marks

(23) Consider the equation $x^2y'' + xy' - y = 0$.

(a) Find a solution y_1 by inspection. 1 Mark

(b) Find an independent solution y_2 of the form vy_1. 3 Marks

(c) Find the general solution. 1 Mark

(24) Find all pairs of C^∞ functions $x(t)$ and $y(t)$ on $\mathbb{R}$ such that $x'(t) = 2x(t) - y(t)$ and $y'(t) = x(t)$.

(Hint: Eliminate y first)