Instructions

(1) This question paper consists of two parts: Part A and Part B and carries a total of 100 Marks.

(2) There is no negative marking.

(3) Part A carries 20 multiple choice questions of 2 marks each. Answer all questions in Part A.

(4) Answers to Part A are to be marked in the OMR sheet provided.

(5) For each question, darken the appropriate bubble to indicate your answer.

(6) Use only HB pencils for bubbling answers.

(7) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(8) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

(9) Part B has 24 questions. Answer any 12 in this part. Each question carries 5 marks.

(10) Answers to Part B are to be written in the separate answer book provided.

(11) Candidates are asked to fill in the required fields on the sheet attached to the answer book.

(12) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the set of integers, real numbers, rational numbers and complex numbers respectively.

(13) For two topological spaces (X_1, τ_1) and (X_2, τ_2), a mapping $f\colon (X_1, \tau_1) \to (X_2, \tau_2)$ is called a homeomorphism if f is a bijection (i.e., one-to-one and onto) and both f and f^{-1} are continuous. If such an homeomorphism exists then (X_1, τ_1) is said to be homeomorphic to (X_2, τ_2) (or (X_1, τ_1) and (X_2, τ_2) are homeomorphic).

(14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denote the standard norm on $\mathbb{R}^n$. The metric given by $d(x, y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

(15) If not mentioned separately then the topology on $\mathbb{R}^n$ is the standard topology (i.e., the topology induced by the standard metric).

(16) $\log(x)$ denotes the logarithm of x to the base e.

MATHEMATICS

PART A

(1) Let $f(x,y) = xy + 2x - \log(x^2 y)$, $x > 0$ and $y > 0$. Then the point $(x,y) = (\frac{1}{2}, 2)$ is a

(A) local minimum.
(B) local maximum but not a global maximum.
(C) saddle point.
(D) global maximum.

(2) Let $\{f_n\}_{n\geq 1}$ be a sequence of functions defined on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} nx, & \text{if } 0 \leq x < \frac{1}{n}, \\ 2 - nx, & \text{if } \frac{1}{n} \leq x < \frac{2}{n}, \\ 0, & \text{if } \frac{2}{n} \leq x \leq 1. \end{cases}$$

Then,

(A) there exists a $x \in [0,1]$ such that the sequence $\{f_n(x)\}_{n\geq 1}$ is not convergent.
(B) the sequence $\{f_n\}_{n\geq 1}$ is equicontinuous.
(C) the sequence $\{f_n\}_{n\geq 1}$ converges uniformly to $f \equiv 0$ on $[0,1]$.
(D) $\int_{[0,1]} |f_n(x)| dx \to 0$, as $n \to \infty$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If $|f'|$ is bounded, then

(A) f is bounded.
(B) $\lim_{x\to\infty} f(x)$ exists.
(C) f is uniformly continuous.
(D) The set $\{x : f(x) = 0\}$ is compact.

(4) Let $f : [0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{x}{1-e^{-x}}, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then the function f is

(A) continuous at $x = 0$.
(B) bounded.
(C) increasing.
(D) zero for at least one $x > 0$.

(5) For $A \subseteq \mathbb{R}^2$, let τ be the subspace topology on $X = \mathbb{R}^2 \setminus A$. Then

(A) A is countable $\Longrightarrow$ (X, τ) is connected.
(B) (X, τ) is connected $\Longrightarrow$ A is finite.
(C) A is unbounded $\Longrightarrow$ (X, τ) is compact.
(D) A is open $\Longrightarrow$ (X, τ) is compact.

(6) Let τ_1 be the topology on $\mathbb{R}$ generated by the base $\mathcal{B} = \{[a, b) : a < b \in \mathbb{R}\}$. If τ_0 is the standard topology on $\mathbb{R}$ and $\mathrm{Id}\colon \mathbb{R} \to \mathbb{R}$ is the identity mapping then

(A) $\mathrm{Id}\colon (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is continuous but not an open mapping.
(B) $\mathrm{Id}\colon (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is an open mapping but not continuous.
(C) $\mathrm{Id}\colon (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is a homeomorphism.
(D) $\mathrm{Id}\colon (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_0)$ is neither continuous nor an open mapping.

(7) Let A, B, C be non-empty sets and $f : A \to B$ and $g : B \to C$ be two mappings. If $g \circ f$ is a bijection, then

(A) both f and g must be bijections.
(B) f must be injective and g must be bijective.
(C) f must be onto and g must be injective.
(D) f must be injective and g must be onto.

(8) If A is a non-empty open subset of $\mathbb{R}^2$, then

(A) A is the interior of its closure.
(B) the closure of A is compact.
(C) A is the countable union of proper open subsets of A.
(D) $f(A)$ is open for any continuous function $f : \mathbb{R}^2 \to \mathbb{R}$.

(9) Let γ be the circle defined by $\gamma(\theta) = 2e^{i\theta}, \quad 0 \le \theta \le 2\pi$. Then, $\int_\gamma \frac{e^z}{z(z-1)}\, dz =$
(A) 2π.
(B) $2\pi i$.
(C) $2\pi i(e+1)$.
(D) $2\pi i(e-1)$.

(10) Consider the function $f(z) = \left(\frac{\sin(e^z - 1)}{z}\right)^2$ on $\mathbb{C} - \{0\}$. Then $z = 0$ is

(A) a simple pole for $f(z)$.
(B) a pole of order 2 for $f(z)$.
(C) an essential singularity for $f(z)$.
(D) a removable singularity for $f(z)$.

(11) The function $f(z) = 1 + 2z + 3z^2 + 4z^3 + \cdots$ is

(A) analytic near $z = 0$.
(B) analytic near $z = 1$.
(C) an entire function.
(D) not analytic near any point in $\mathbb{C}$.

(12) The set of complex numbers $z = x + iy$ such that $|e^{z^2}| \le 1$ is given by

(A) $-x \le y \le x$.
(B) $-y \le x \le y$.
(C) $-x^2 \le y \le x^2$.
(D) $-y^2 \le x \le y^2$.

(13) Let A, B be two real 3×3 matrices. Then,

(A) $\text{Rank}(AB) = \min\{3, \text{Rank}(A) + \text{Rank}(B)\}$.
(B) $\text{Rank}(AB) = \min\{3, \text{Rank}(A) \times \text{Rank}(B)\}$.
(C) $\text{Rank}(AB) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}$.
(D) $\text{Rank}(AB) \geq \max\{\text{Rank}(A), \text{Rank}(B)\}$.

(14) Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix}.$$

The dimension of the null space $\mathcal{N}(A)$ is

(A) 0.
(B) 1.
(C) 2.
(D) 3.

(15) Let K be a non-zero real 3×3 matrix. If K is skew symmetric, then

(A) K is invertible.
(B) all eigen values of K are real.
(C) e^{tK} is an orthogonal matrix for $t \in \mathbb{R}$.
(D) $\text{trace}(K) > 0$.

(16) Which of the following is the order of a non-abelian group?

(A) 4.
(B) 8.
(C) 9.
(D) 13.

(17) The number of subgroups of the cyclic group G of order 15, excluding the trivial group and G, is

(A) 2.
(B) 3.
(C) 13.
(D) 14.

(18) Let S_4 be the group of permutations on four letters. The number of elements of order 2 in the group S_4 is

(A) 6.
(B) 9.
(C) 4.
(D) 12.

(19) Let $R[x]$ be the ring of polynomials in one variable. For which one of the following polynomials $f(x)$, is the quotient ring $R/\langle f(x)\rangle$ a field?

(A) $x^3 - 1$.
(B) $x^3 + 1$.
(C) $x^2 - 2$.
(D) $x^2 + 2$.

(20) The general solution of the ODE: $xy' + y = 1$ is

(A) $y = (1 + Cx)^{-1}$.
(B) $y = C + x^{-1}$.
(C) $y = C(1 + x^{-1})$.
(D) $y = 1 + Cx^{-1}$.

PART B

(1) Consider the real sequence $\{a_n\}_{n\geq 2}$ defined by

$$a_n = n(a\log(n) + b\log(\log(n)))e^{-[a\log(n)+b\log(\log(n))]}.$$

Find all real values of a and b for which the series $\sum_{n\geq 2} a_n$ converges.

(2) Let $f, u : [0, \infty) \to [0, \infty)$ be such that u is onto, and f is continuous. If $g(x) = \int_0^{u(x)} f(t)dt$ is constant for all x, then show that $f \equiv 0$.

(3) Find the maximum and minimum of the function $f(x, y, z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$.

(4) Consider the polynomial $P(x) = \sum_{j=0}^{n} c_j x^j$, where n is a non-negative odd integer and $c_0, c_n \neq 0$. If $c_0 c_n > 0$, then show that P has at least one zero on the negative half of the real line.

(5) If a is a positive constant, then show that $\lim_{N\to\infty} \prod_{n=1}^{N} (1 - e^{-na})$ exists and is strictly positive.

[Here $\prod$ denotes the product.]

(6) Let f be a measurable function on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} |f(x)|dm(x) < \infty$, where m is the Lebesgue measure on the Borel subsets of $\mathbb{R}$. Show that the function $g(t) = \int_{\mathbb{R}} e^{itx} f(x)dm(x)$ is uniformly continuous on $\mathbb{R}$.

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that there is **no** $x \in \mathbb{R}$ for which $f(x) = 0 = f'(x)$. Show that the set $S = \{x : \ f(x) = 0, 0 \leq x \leq 1\}$ is a finite set.

(8) Let (X, τ) be a Hausdorff topological space. If X is finite then show that (X, τ) is metrizable (i.e., τ can be obtained from a metric d on X).

(9) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a bijective mapping. If f is continuous then show that f is a homeomorphism.

(10) Show that $[0,1] \times [0,1)$ is homeomorphic to $[0,1) \times [0,1)$.
[Here the topologies are the subspace topologies from $\mathbb{R}^2$.]

(11) Let Δ denote the open unit disc in the complex plane and $f : \Delta \to \mathbb{C}$ be analytic. Suppose that $f''(\frac{1}{2^n}) = f(\frac{1}{2^n})$ for all $n = 1, 2, \ldots$. Show that there exists an entire function $F : \mathbb{C} \to \mathbb{C}$ such that $F(z) = f(z)$ for all $z \in \Delta$.

(12) Find $c \in \mathbb{C}$ so that the function $f(z) = \frac{2z^2+1}{z^4-5z^2+4} - \frac{c}{z+1}$ is analytic near $z = -1$.

(13) Let Δ be the unit disc in the complex plane. Show that there is no non-constant analytic function $f : \Delta \to \mathbb{C}$ that satisfies

$$\mathrm{Im}(f(z)) = \sin(\mathrm{Re}(f(z)) + \mathrm{Re}(f(z)) \qquad \text{for all} \quad z \in \Delta.$$

(14) Let U and V be two subspaces of a finite dimensional inner product space $(W, \langle \cdot, \cdot \rangle)$. If $W = U \oplus V$ (V is not necessarily $U^\perp$) then show that $W = U^\perp \oplus V^\perp$.
[Here $V^\perp := \{\alpha \in W \ : \ \langle \alpha, \beta \rangle = 0, \text{ for all } \beta \in V\}$.]

(15) Consider the mapping $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x,y,z) = (x-y, y-z, z-x)$. Show that f is linear. Find the image of f.

(16) For $i = 1, 2, 3$, let P_i be the plane in $\mathbb{R}^3$ given by $a_i x + b_i y + c_i z + d_i = 0$. If the rank of the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is 2 then show that either $P_1 \cap P_2 \cap P_3 = \emptyset$ (empty set) or $P_1 \cap P_2 \cap P_3$ is a straight line.

(17) Let θ denote the 3×3 null-matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Let $A \neq \theta$ be a 3×3 real matrix. If $A^3 + A = \theta$, then show that the rank of A is 2.

(18) Let A be a 2×2 matrix such that $A^3 = \theta$, where θ is the 2×2 null matrix. Show that $A^2 = \theta$.

(19) Suppose G is an abelian group of order 105. Show that G is cyclic.

(20) Let G be a group of order pq, with p and q distinct prime numbers. If there is an element g (not equal to the identity element) in the group G, with the property that $gh = hg$ for all $h \in G$, then show that the group G is abelian.

(21) Let $\mathbb{R}[x, y]$ be the ring of polynomials in two variables over the real numbers and let $f(x)$ and $g(y)$ be non-constant polynomials. Show that the ideal generated by $f(x)$ and $g(y)$ is not a principal ideal.

(22) Find all the group homomorphisms from the group of rational numbers $(\mathbb{Q}, +)$ to the group of integers $(\mathbb{Z}, +)$.

(23) If y_1 and y_2 are two linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

then show that

$$P(x) = -\frac{y_1 y_2'' - y_2 y_1''}{W(y_1, y_2)}$$

and

$$Q(x) = \frac{y_1' y_2'' - y_2' y_1''}{W(y_1, y_2)},$$

where W is the Wronskian.

(24) Solve

$$y' = (\cos x)(y - 2), \qquad y(0) = 1.$$

Find the unique solution of the same equation with the initial condition $y(0) = 2$. Specify the interval on which the solution is defined in each case.