Instructions

- (1) This question paper consists of two parts: Part A and Part B and carries a total of 100 Marks.
- (2) There is no negative marking.
- (3) Candidates are asked to fill in the required fields on the sheet attached to the answer book.
- (4) Part A carries 20 multiple choice questions of 2 marks each. Answer all questions in Part A.
- (5) Answers to Part A are to be marked in the OMR sheet provided.
- (6) For each question, darken the appropriate bubble to indicate your answer.
- (7) Use only HB pencils for bubbling answers.
- (8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.
- (9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.
- (10) Part B has 12 questions. Answer any 6 in this part. Each question in this part carries 10 marks.
- (11) Answers to Part B are to be written in the separate answer book provided.
- (12) Answer to each question in Part B should begin on a new page.
- (13) Let \mathbb{Z} , \mathbb{R} , \mathbb{Q} and \mathbb{C} (\mathbb{Z}_+ , \mathbb{R}_+ , \mathbb{Q}_+ and \mathbb{C}_+) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.
- (14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \dots, x_n)\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$ denote the standard norm on \mathbb{R}^n . The metric given by $d(x, y) = \|x y\|$ is called the standard metric on \mathbb{R}^n .

MATHEMATICS

PART A

- (1) The limit, $\lim_{R\to\infty} \frac{\int_R^{\infty} r^n e^{-\frac{r^2}{2}} dr}{R^{n-1}e^{-\frac{R^2}{2}}}$, $n\in\mathbb{Z}_+$, equals
 - (A) -1.
 - (B) 0.
 - (C) 1.
 - (D) ∞ .
- (2) The function $f: \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) f(y)| \le c|x y|$ for all $x, y \in \mathbb{R}$ and some constant $c \in \mathbb{R}_+$. Then,
 - (A) f must be bounded.
 - (B) f must be continuous but may not be uniformly continuous.
 - (C) f must be uniformly continuous but may not be differentiable.
 - (D) f must be differentiable.
- (3) Consider the sequence $\{x_n\}_{n\geq 1}$ defined recursively as $x_1=1$,

$$x_n := \sup \left\{ x \in [0, x_{n-1}) : \sin \left(\frac{1}{x}\right) = 0 \right\}, \qquad n \ge 2.$$

Then, $\limsup_{n\to\infty} x_n$ equals

- $(A) -\infty.$
- (B) 0.
- (C) 1.
- (D) ∞ .
- (4) Let $f:[0,1]\to\mathbb{R}$ be continuous. Then $\int_0^1 f(x)e^{-x}dx$ equals
 - (A) $f'(0) f'(1)e^{-1}$.
 - (B) $\int_0^1 e^{-x} (\int_0^x f(y) \, dy) \, dx$.
 - (C) $f(c)(1-e^{-1})$, for some $c \in [0,1]$.
 - (D) $e^{-c} \int_0^1 f(x) dx$, for some $c \in [0, 1]$.

- (5) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous with f(0) = f(1) = 0. Which of the following is **not** possible?
 - (A) $f([0,1]) = \{0\}.$
 - (B) f([0,1]) = [0,1).
 - (C) f([0,1]) = [0,1].
 - (D) $f([0,1]) = [-\frac{1}{2}, \frac{1}{2}].$
- (6) The series $\sum_{n=1}^{\infty} \frac{e^{i\sqrt{n}x}}{n^2}$ converges
 - (A) only at x = 0.
 - (B) only for $|x| \leq 1$.
 - (C) converges pointwise for all $x \in \mathbb{R}$, but not uniformly.
 - (D) converges uniformly on \mathbb{R} .
- (7) Let P(x) be a non-zero polynomial of degree N. The radius of convergence of the power series

$$\sum_{n=0}^{\infty} P(n)x^n$$

- (A) depends on N.
- (B) is 1 for all N.
- (C) is 0 for all N.
- (D) is ∞ for all N.
- (8) The function $f(z) = \exp\left(\left(\frac{\cos z 1}{z^2}\right)^2\right)$
 - (A) has a removable singularity at z = 0.
 - (B) has a pole of order 2 at z = 0.
 - (C) has a pole of order 4 at z = 0.
 - (D) has an essential singularity at z = 0.

- (9) The region described by $\left|\frac{z-i}{z+i}\right| < 1$, where $z = x+iy \in \mathbb{C}$ is
 - (A) $\{z \in \mathbb{C} : x < 0\}$.
 - (B) $\{z \in \mathbb{C} : x > 0\}.$
 - (C) $\{z \in \mathbb{C} : y < 0\}.$
 - (D) $\{z \in \mathbb{C} : y > 0\}.$
- (10) Let $A \neq \mathbb{R}$ be a dense subset of \mathbb{R} . If $U \subseteq \mathbb{R}$ is a non-empty open subset then
 - (A) $U \subseteq \overline{A \cap U}$.
 - (B) $\overline{A \cap U} = \emptyset$.
 - (C) $\overline{A \cap U} \subseteq U$.
 - (D) $\overline{A \cap U} = A \cap \overline{U}$.
- (11) Let $X=\{1/n:n\in\mathbb{Z},n\geq 1\}$ and let \bar{X} be its closure. Then
 - (A) $\bar{X} \setminus X$ is a single point.
 - (B) $\bar{X} \setminus X$ is open in \mathbb{R} .
 - (C) $\bar{X} \setminus X$ is infinite but not open in \mathbb{R} .
 - (D) $\bar{X} \setminus X = \phi$.
- (12) Let X be a subset of \mathbb{R} homeomorpic to $(0,\pi)$. Then
 - (A) X must be bounded.
 - (B) X must be compact.
 - (C) The closure \bar{X} of X must be unbounded.
 - (D) X must be open.

- (13) Let $A \subseteq \mathbb{R}$ be an open set. If $(0,1) \cup A$ is connected, then
 - (A) A must be connected.
 - (B) A must have one or two componets.
 - (C) $A \setminus (0,1)$ has at most two components.
 - (D) A must be a Cantor set.
- (14) Let A be an $n \times n$ matrix with entries 0 and 1 and n > 1. If there is exactly one non zero entry in each row and each column of A, then the determinant of A must be
 - (A) ± 1 .
 - (B) 0.
 - (C) n.
 - (D) 1.
- (15) Let A be an $n \times n$ matrix over real numbers such that AB = BA for all $n \times n$ matrices B. Then
 - (A) A must be 0.
 - (B) A must be the identity.
 - (C) A must be a diagonal matrix.
 - (D) A must be either 0 or the identity.
- (16) Let A be a matrix such that $A^3 = -I$. Then which of the following numbers can be an eigenvalue of A?
 - (A) i.
 - (B) 1.
 - (C) -1.
 - (D) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

- (17) Let V be the vector space of all polynomials whose degree is less than or equal to n. Let $D:V\to V$ be the differentiation operator on V, that is, DP(x)=P'(x). Then the trace of D, $\mathrm{tr}(D)$ equals
 - (A) 0.
 - (B) 1.
 - (C) n.
 - (D) n^2 .
- (18) Let A be a 3×3 matrix over real numbers satisfying $A^{-1} = I 2A$. Then the determinant of A, det(A) equals
 - (A) $-\frac{1}{2}$.
 - (B) $\frac{1}{2}$.
 - (C) 1.
 - (D) 2.
- (19) For which of the following integers n is every group of order n abelian?
 - (A) n = 6.
 - (B) n = 9.
 - (C) n = 12.
 - (D) n = 18.
- (20) Let G be the cyclic subgroup of order 18. The number of subgroups of G, including G and the trivial group, is
 - (A) 4.
 - (B) 6.
 - (C) 9.
 - (D) 18.

PART B

- (1) Let $F_n: \mathbb{R} \to [0,1], n \geq 0$, be continuous functions satisfying
 - (i) $F_n(x) \leq F_n(y)$ for all $x \leq y$,
 - (ii) $\lim_{x \to -\infty} F_n(x) = 0$, and (iii) $\lim_{x \to \infty} F_n(x) = 1$.

Suppose that F_n converges pointwise to F_0 on \mathbb{R} , that is $F_n(x) \to F_0(x)$ for all $x \in \mathbb{R}$, as $n \to \infty$. Show that F_n converges uniformly to F_0 on \mathbb{R} , as $n \to \infty$.

(2) Let $f:[0,1]\to\mathbb{R}$ be a continuous function of bounded variation. Then for any $p \in (1, \infty)$, show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left| f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right|^{p} = 0.$$

- (3) Given n points z_1, z_2, \dots, z_n on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, prove that there exists a point z on the unit circle such that $\prod_{i=1}^{n} |z-z_i| \ge 1$.
- (4) Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function. Let $\Delta(0, R) = \{z = x + iy \in \mathbb{C} : |z| < R\},\$ denote the open disc in the complex plane around the origin of radius R > 0. If $n-1 \in \mathbb{Z}_+$, show that

$$\int_{\Delta(0,R)} \overline{z}^{n-1} f(z) \ dx \ dy = \frac{\pi R^{2n}}{n!} f^{(n-1)}(0),$$

where $f^{(k)}(0)$ denotes the k-th derivative of f at the origin.

(5) Let $f: \Delta(0,1) \to \mathbb{C}$ be analytic. Show that it is not possible for f(z) to satisfy

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{n}{n+1},$$

for all $n \geq 2$.

(6) For $A, B \subseteq \mathbb{R}^2$, define the distance $d(A, B) := \inf\{||x - y|| : x \in A, x \in B\}$. Let $C, D \subseteq \mathbb{R}^2$ be two closed subsets. If $C \cap D = \emptyset$ and d(C, D) = 0 then show that both C and D are unbounded.

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- (7) Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation and let $V \subset \mathbb{R}^n$ be a subspace such that $L(V) \subseteq V$. Show that L(V) = V. Here $L(V) = \{L(v) : v \in V\}$.
- (8) Let A be a 2×2 invertible matrix over real numbers such that for some 2×2 invertible matrix P, $PAP^{-1} = A^2$. Show that either $A^3 = I$ or $QAQ^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, for some invertible 2×2 matrix Q.
- (9) Suppose V is an n-dimensional vector space over a field F. Let $W \subseteq V$ be a subspace of dimension r < n. Show that $W = \cap \{U : U \text{ is an } (n-1)\text{-dimensional subspace of } V \text{ and } W \subseteq U\}$.
- (10) Suppose G is a finite group. Show that every element x of G can be expressed as $x = y^2$ for some $y \in G$ if and only if the order of G is odd.
- (11) Let G be a group with identity e. Let N_1, N_2, N_3 be three normal subgroups of G. If $N_i \cap N_j = \{e\}$ and $N_i N_j = G$ for $1 \le i \ne j \le 3$ then show the following:
 - (i) xy = yx for $x \in N_i$, $y \in N_j$, $1 \le i \ne j \le 3$.
 - (ii) yz = zy for $y, z \in N_i$, $1 \le i \le 3$.
 - (iii) G is commutative.

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(12) Let $y: \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function which satisfies

$$y'' + y' - y > 0$$
, $y(0) = y(1) = 0$.

If $y(x) \ge 0$ for all $x \in [0, 1]$, prove that y is identically zero in [0, 1].