MATHEMATICS March, 2008

Answer *all* the *ten* questions : $10 \times 1 = 10$

1. Find the least positive integer x satisfying $2x + 5 \equiv x + 4 \pmod 5$.

2. If $A = \begin{bmatrix} 5 - x & 2y - 8 \\ 0 & 3 \end{bmatrix}$ is scalar matrix, find x and y.

3. If $a * b = \dfrac{3ab}{7}$, then prove that $*$ is associative.

4. Define co-planar vectors.

5. Write the condition for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches both axes.

6. Find the co-ordinates of the end points of length of the latus rectum of the parabola $y^2 = 12x$.

7. Find the value of $\tan(\tan^{-1} 3) + \sec^{-1}\{\sec(-2)\}$.

8. Write the multiplicative inverse of i.

9. Define the differential coefficient of a continuous function $y = f(x)$ w.r.t. x.

10. Evaluate $\displaystyle\int \frac{1 - \cos x}{\sin^2 x}\, dx$.

PART – B

Answer any *ten* questions : $10 \times 2 = 20$

11. The relation 'Congruence modulo m' is an equivalence relation on z or prove that $a \equiv b \pmod m$ is an equivalence relation on z.

12. Evaluate $\begin{vmatrix} 2001 & 2004 \\ 2007 & 2010 \end{vmatrix}$.

13. If in a group $(G, *)$ $\forall\, a \in G$, $a^{-1} = a$, then prove that $(G, *)$ is an Abelian group.

14. If the vectors $\lambda \hat{i} + 2\hat{j} - \hat{k}$ and $\hat{i} - 3\hat{j} + 2\hat{k}$ are orthogonal, find λ.

15. Find the area of the circle whose parametric equations are

$x = 3 + 2\cos\theta$ and $y = 1 + 2\sin\theta$.

16. Find the equation of the hyperbola in the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Given that transverse axis = 10, and eccentricity (e) = 2.

17. Find x if $\tan^{-1} = \sin^{-1}\dfrac{1}{2} + \cos^{-1}\dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

18. Prove that $e^{1 + i\pi/3} + e^{1 - i\pi/3} = e$.

19. If $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$, then find $\dfrac{dy}{dx}$ at (a, b).

20. Find the length of the sub-tangent to the curve $x^3 + xy + y^2 = 13$ at $(1, 3)$.

21. Evaluate $\int \frac{1}{\sin^2 x \cos^2 x} \, dx$.

22. Form the differential equation by eliminating the parameter c.

$\sin^{-1} x + \sin^{-1} y = c$.

PART – C

I. Answer any *three* questions : $3 \times 5 = 15$

23. Find the number of all positive divisors and the sum of all positive divisors of 39744. 5

24. a) Show that

$$\begin{vmatrix} a^2 + bc & a & 1 \\ b^2 + ca & b & 1 \\ c^2 + ab & c & 1 \end{vmatrix} = -2(a-b)(b-c)(c-a).$$

b) Find the values of x and y according to Cramer's rule :

$x + 2y = 7$

$4x - 5y = 2$. 2

25. a) Prove that the set $H = \{1, 2, 4\} \otimes_7$ is a sub-group of the group
$G = \{1, 2, 3, 4, 5, 6\} \otimes_7$ under multiplication modulo 7. 3

b) Prove that the identity element of a group is unique. 2

26. a) If the vectors $\hat{i} - \hat{j} + \lambda\hat{k}$, $4\hat{i} + 2\hat{j} + 9\hat{k}$, $5\hat{i} + \hat{j} + 14\hat{k}$ and $3\hat{i} + 2\hat{j} + 7\hat{k}$ are the position vectors of the four coplanar points, find λ. 3

b) Find the unit vector in the direction of $2\hat{i} - \hat{j} + 2\hat{k}$. 2

II. Answer any *two* questions : $2 \times 5 = 10$

27. a) Find the equation of the circle which cuts the two circles $x^2 + y^2 - 6y + 1 = 0$ and $x^2 + y^2 - 4y + 1 = 0$ orthogonally and whose centre lies on the line $3x + 4y + 5 = 0$. 3

b) Find the equation of the circle having $(4, 2)$ and $(-5, 7)$ as end points of the diameter. 2

28. a) Find the condition for the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. 3

b) Find the focus of the parabola $y^2 - 8x - 32 = 0$. 2

29. Prove that

$$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}} = 0$$ 5

III. Answer any *three* of the following questions : $3 \times 5 = 15$

30. a) Differentiate $\operatorname{cosec}(ax)$ w.r.t. x from the first principle. 3

b) Differentiate $\sin x$ with respect to $\log_e x$. 2

31. a) If $e^x + e^y = e^{x+y}$ prove that $\frac{dy}{dx} = -e^{y-x}$. 2

b) If $x = \tan^{-1}\sqrt{\frac{1-t}{1+t}}$, $y = \cos^{-1}(4t^3 - 3t)$, prove that

$$\frac{dy}{dx} = 6.$$ 3

32. a) If $y = \sin^2 \left\{ \cot^{-1} \sqrt{\frac{1+x}{1-x}} \right\}$, prove that $\frac{dy}{dx} = -\frac{1}{2}$. 3

b) Evaluate $\int \frac{\sin x}{1 + \sin x}\, dx$. 2

33. a) Evaluate $\int \frac{\cos x}{2\sin^2 x + 3\sin x + 4}\, dx$. 3

b) Evaluate $\int \frac{x}{\sqrt{x^2 - 4}}\, dx$. 2

34. Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ by integration method. 5

PART – D

Answer any *two* of the following questions : 2 × 10 = 20

35. a) Define an ellipse. Derive the equation of the ellipse in the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ 6

b) If $A = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$, find A^{-1} by Cayley-Hamilton theorem. 4

36. a) State and prove D'Moivre's theorem for rational index. 6

b) Prove that the sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ by vector method. 4

37. a) Prove that the greatest size rectangle that can be inscribed in a circle of radius a is a square. 6

b) Find the general solution of

$(\sqrt{3}+1)\cos\theta + (\sqrt{3}-1)\sin\theta = 2.$ 4

38. a) Prove that $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$. 6

b) Solve the differential equation $\frac{dy}{dx} = (x+y-1)^2$. 4

PART - E

Answer any *one* of the following questions : $1 \times 10 = 10$

39. a) Find the cube roots of $1+i$ and represent the Argand diagram. 4

b) Find the length of the chord intercepted by the circle

$x^2 + y^2 - 8x - 6y = 0$ and the line $x - 7y - 8 = 0$. 4

c) Find the digit in the unit place of 7^{123}. 2

40. a) If $|\vec{a}| = 13$, $|\vec{b}| = 19$, $|\vec{a}+\vec{b}| = 24$, find $|\vec{a}-\vec{b}|$. 4

b) Find $\int \tan^4 x \, dx$. 4

c) If $y = \log\sqrt{\cos x}$, find $\frac{dy}{dx}$. 2