MATHEMATICS - JUNE 2008

PART – A

Answer *all* the *ten* questions. 10 · 1 = 10

1. Find the number of incongruent solutions of $9x \equiv 21 \pmod{30}$.

2. Evaluate $\begin{vmatrix} 4321 & 4322 \\ 4323 & 4324 \end{vmatrix}$.

3. In a group $(Z_6, +\text{mod } 6)$, find $2 +_6 4^{-1} +_6 3^{-1}$.

4. Find the position vector of the point P which is the mid-point AB where the position vectors of A and B are $\hat{i} + \hat{j} + 2\hat{k}$ and $3\hat{i} - 3\hat{j} + 2\hat{k}$.

5. Find the equation to a circle whose centre is $(a, 0)$ and touching the y-axis.

6. Find the equation to directrix of $(x+1)^2 = -4(y-3)$.

7. Find the value of $\cos^{-1}(\sin 330^\circ)$.

8. If $1, \omega, \omega^2$ are the cube roots of unity, find the value of $(1 + \omega - \omega^2)$.

9. If $y = e^{\sqrt{x}} + x^{\sqrt{e}}$, find $\frac{dy}{dx}$.

10. Evaluate $\int e^x \left(\frac{1 + \tan x}{\cos x} \right) dx$.

PART – B

Answer any *ten* questions. 10 × 2 = 20

11. Find the G.C.D. of 352 and 891.

12. Find the characteristic roots of the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

13. Prove that a group of order three is Abelian.

14. Find the volume of the parallelopiped whose co-terminus edges are the vectors $\hat{i} + 3\hat{j} + 2\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$.

15. Find the equation to the parabola whose focus is (3, 2) and its directrix is $x = 1$.

16. Prove that

$$\sin \left[2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right] = \sqrt{1-x^2}.$$

17. Find the equation of a circle passing through the origin, having its centre on the line $y = x$ and cutting orthogonally the circle

$x^2 + y^2 - 4x - 6y + 10 = 0.$

18. Prove that $(1-i)^9 = 16 - 16i$.

19. If $y = \log_e \left(\frac{1 - \cos x}{1 + \cos x} \right)$, then prove that $\frac{dy}{dx} = 2 \operatorname{cosec} x$.

20. Find the point on the curve $y^2 = x$ the tangent at which makes an angle of 45° with the x-axis.

21. Evaluate $\int_0^1 x(1-x)^7\, dx$.

22. Form the differential equation by eliminating the arbitrary constant

$(y-2)^2 = 4a(x+1)$.

PART – C

I. Answer any *three* questions : $3 \times 5 = 15$

23. a) Find the number of positive divisors and sum of all such positive divisors of 756. 3

b) If a / bc and $(a, b) = 1$, then prove that a / c. 2

24. Solve by matrix method :

$$3x + y + 2z = 3$$

$$2x - 3y - z = -3$$

$$x + 2y + z = 4.$$

25. Prove that the set z of integers is an Abelian group under binary operation $*$ defined by $a * b = a + b + 3, \ \forall a, b \in z$. 5

26. a) If $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - 2\hat{k}$, find a unit vector perpendicular to $\vec{a}$ and in the same plane on $\vec{b}$ and $\vec{c}$. 3

b) Find the area of a parallelogram whose diagonals are the vectors

$2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} - 2\hat{j} + 3\hat{k}$. 2

II. Answer any *two* questions : $2 \times 5 = 10$

27. a) Find the length of the tangent from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. 3

b) Find the equations of tangent to the circle

$x^2 + y^2 - 2x - 4y - 4 = 0$, which are perpendicular to

$3x - 4y + 6 = 0$. 2

28. a) Find the focus and equation to the directrix of the ellipse

$9x^2 + 5y^2 - 36x + 10y - 4 = 0$. 3

b) Find the equation to the hyperbola in the standard form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, given that length of latus rectum $= \frac{14}{3}$ and

$e = \frac{4}{3}$. 2

29. a) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, prove that

$xy + yz + zx = 1$. 3

b) Find the general solution of $\sin^2 \theta - \cos 2\theta = \frac{5}{4}$. 2

III. Answer any *three* of the following questions : $3 \times 5 = 15$

30. a) Differentiate a^x w.r.t. x by first principles. 3

b) If $y = \tan^{-1}\left(\frac{4x}{4 - x^2}\right)$, prove that $\frac{dy}{dx} = \frac{4}{4 + x^2}$. 2

31. a) If $y = (\sin^{-1} x)^2 + (\cos^{-1} x)^2$, prove that

$(1 - x^2)\, y_2 - x y_1 - 4 = 0.$ 3

b) If $x = 3 \sin 2\theta + 2 \sin 3\theta$, and

$y = 2 \cos 3\theta - 3 \cos 2\theta$,

prove that $\frac{dy}{dx} = -\tan\frac{\theta}{2}$. 2

32. a) Prove that in the curve $y = e^{\overline{a}}$ the subnormal varies as the

square of the ordinate and subtangent is constant. 3

b) Evaluate $\int_0^{\pi/2} \frac{\sin x \, . \, \cos x}{1 + \sin^4 x}\, dx$. 2

33. a) Evaluate $\int \frac{2 - 3 \tan x}{1 + 2 \tan x}\, dx$. 3

b) Evaluate $\int \frac{1}{(1 + e^x)(1 - e^{-x})}\, dx$. 2

34. Find the area of the ellipse $9x^2 + 16y^2 = 144$ by integration. 5

PART – D

Answer any *two* of the following questions : $2 \times 10 = 20$

35. a) Define hyperbola as a locus and derive the standard equation of the hyperbola in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. 6

b) Prove that $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (a^3 - 1)^2$. 4

36. a) If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, prove that

i) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$. 6

b) Prove that $[\vec{a} \times \vec{b} \;\; \vec{b} \times \vec{c} \;\; \vec{c} \times \vec{a}] = [\vec{a} \;\; \vec{b} \;\; \vec{c}]^2$. 4

37. a) The surface area of a sphere is increasing at the rate of 8 sq.cm/sec. Find the rate at which the radius and the volume of the sphere are increasing when the volume of the sphere is $\frac{500\pi}{3}$ c.c. 6

b) Find the general solution of $\sin\theta + \sin 2\theta + \sin 3\theta = 0$. 4

38. a) Prove that $\int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x \cos x}\, dx = \frac{\pi}{3\sqrt{3}}$. 6

b) Find the general solution of the differential equation

$$xy \frac{dy}{dx} = \frac{1 + y^2}{1 + x^2} (1 + x + x^2) .$$ 4

PART – E

Answer any *one* of the following questions : 1 × 10 = 10

39. a) If $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$. 4

b) Find the cube roots of a complex number $\sqrt{3} - i$ and represent them in argand diagram. 4

c) Find the remainder when 2^{202} is divided by 11 (least positive remainder). 2

40. a) The sum of the lengths of a hypotenuse and another side of a right angled triangle is given. Show that the area of the triangle is maximum when the angle between these sides is $\frac{\pi}{3}$. 4

b) Evaluate $\int \cot^4 (3x)\, dx$. 4

c) Differentiate w.r.t. x :

$$y = \log_5 \sqrt{1 - x^2} .$$ 2