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M.Sc. DEGREE EXAMINATION, MAY 2011

Second Year

Mathematics

Paper I — TOPOLOGY AND FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 100 marks

Answer any FIVE questions choosing atleast TWO from each part.

PART A

1. (a) Let X be a topological space. Then prove that any closed subset of X is the disjoint union of its set of isolated points and its set of limit points, in the sense that it contains these sets, they are disjoint and it is their union.

   (b) Define a 'Topology' on a non empty set X. Let X be a second countable space. Then prove that any open base for X has a countable sub class which is also an open base.

2. (a) Prove that any continuous image of a compact space is compact.

   (b) State and prove the Heine-Borel theorem.

3. (a) State and prove Tychonoff's theorem.

(b) Prove that a closed subspace of a complete metric space is compact Û it is totally bounded.

4. (a) Prove that every compact Hausdorff space is normal.

(b) State and prove Urysohn's Lemma.

5. (a) Prove that the product of any non-empty class of connected spaces is connected.

(b) Prove that a one-to-one continuous mapping of a compact space onto Hausdorff space is a homomorphism.

PART B

6. (a) If N is a normed linear space, then prove that the closed unit sphere S* in N* is a compact Hausdorff space in the weak* topology.

(b) Let N be a nonzero normed linear space, prove that N is a Banach space if and only if

$\{x \hat{I}\ N / \|x\| = 1\}$ is complete.

7. (a) State and prove the open mapping theorem.

(b) If N is a normed linear space and x_0 is a non-zero vector in N, then prove that there exists a functional to in N* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

8. (a) Define a Hilbert space. Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

   (b) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M$ Å M^{1}.

9. (a) State and prove Bessel's inequality.

   (b) If T is an operator on H for which $(Tx, x) = 0$ for all x, then prove that $T = 0$.

10. (a) If N is a normal operator on H, then prove that $\|N^2\| = \|N\|^2$.

    (b) Prove that a closed linear subspace M of H is invariant under an operator T Û $M^{\wedge}$ is invariant under T*.

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M.Sc. DEGREE EXAMINATION, MAY 2011.

Second Year

Mathematics

Paper II —MEASURE AND FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) State the axiom of Archimedes. Show that between any two real numbers x and y there is a rational number r such that $x < r < y$.

   (b) Show that inf E < Sup E if and only $E \neq \phi$.

2. (a) Define a measurable set. If E_1 and E_2 are measurable, show that $E_1 \cup E_2$ is also measurable.

   (b) Prove that every borel set is measurable. In particular each open set and each closed set is measurable.

3. (a) Let C be a constant and f and g two measurable real-valued functions define on the same domain. Then show that the functions $f+c, cf, f+g, g-f,$ and fg are also measurable.

(b) Let $\{A_n\}$ be a countable collections of sets of real numbers. Then show that $m^*(\cup A_n) \leq \sum m^* A_n$.

4. (a) State and prove Bounded convergence theorem.

(b) Let $<fn>$ be a sequence of measurable functions that converges in measure to f. Then prove that there is a subsequence $<f_{nk}>$ that converges to f almost everywhere.

5. (a) State and prove lebesgue convergence theorem.

(b) Let f be a bounded function defined on [a, b]. If f is Riemann integrable on [a, b], then show that it is measurable and

$$R\int_a^b f(x)\, dx = \int_a^b f(x)\, dx \quad .$$

6. (a) Let f be an integrable function on [a, b] and suppose that

$$F(x) = F(a) + \int_a^x f(t)\,dt .$$

Then show that $f'(x) = f(x)$ for almost all x in [a, b].

(b) Prove that a function F is an indefinite integral if and only if it is absolutely continouous.

7. (a) Prove that the $\lfloor p$ spaces are complete.

(b) State and prove Holder inequality.

8. (a) State and prove Hahn decomposition theorem.

(b) If $\in_i \in B$, then show that

$$\mu\left(\bigcup_{i=1}^{\infty} Ei\right) \leq \sum_{i=1}^{\infty} \mu\, Ei \, .,$$

9. State and prove that Radon – Nikodym theorem.

10. (a) Prove that the set function μ^* is an outer measure.

(b) Define a measure on n algebra. Show that with the usual notation the class B of μ^* measurable set is a σ- algebra.

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M.Sc. DEGREE EXAMINATION, MAY 2011.

Second Year

Mathematics

Paper III — ANALYTICAL NUMBER THEORY AND GRAPH THEORY

Time : Three hours Maximum : 100 marks

Answer any FIVE out of Ten questions selecting atleast TWO from each Part.

All questions carry equal marks.

PART A

1. Show that for all $x \geq 1$, we have

$$\sum_{n \leq x} d(x) = x \log x + (2c - 1)x + O(\sqrt{x})$$ where cis Euler's constant.

2. (a) Prove that two lattice points (a, b) and (m, n) are mutually visible if and only if $a - m$ and $b - n$ are relatively prime.

(b) Show that for all $x \geq 1$, we have

$$\left| \sum_{n \leq x} \frac{u(n)}{n} \right| \leq 1,$$

with equality holding only if $x < 2$.

3. (a) Prove that for $n \geq 1$, the n[th] prime P_n satisfies the inequalities.

$$\frac{1}{6} n \log n < P_n < 12\left(n \log n + n \log \frac{12}{e}\right).$$

(b) State and prove Abel's identity.

4. (a) Prove that there is a constant A such that $\sum_{p \leq x} \frac{1}{p} = \log \log x + A + O\left(\frac{1}{\log x}\right)$ for all $x \geq 2$.

(b) Let F be a real-or complex-valued function defined on $(0, \infty)$ and let

$$G(x) = \log x \sum_{n \leq x} F\left(\frac{x}{n}\right)$$

Then show that

$$F(x) \log x + \sum_{n \leq x} \left(\frac{x}{n}\right) \wedge (n) = \sum_{d \leq x} \mu(d) G\left(\frac{x}{d}\right).$$

PART B

5. (a) Prove that a connected graph G is an Euler graph if and only if can be decomposed into circuits.

(b) Prove that if a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.

6. (a) Prove that a simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

(b) Prove that a sufficient condition for a simple graph G to have a Hamiltonian circuit is that the degree of every vertex in G be at least $\frac{n}{2}$, where n is the number of vertices in G.

7. (a) Prove that any connected graph with n vertices and $n-1$ edges is a tree.

(b) Define Spanning tree. Prove that every connected graph has at least one spanning tree.

8. (a) Prove that every circuit has an even number of edges in common with any cut-set.

(b) Prove that a vertex v in a connected graph G is a cut-vertex if and only if there exist two vertices x and y in G such that every path between x and y passes through v.

9. (a) Prove that the complete graph of five vertices is nonplanar.

(b) Prove that any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

10. (a) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge-disjoint union of circuits.

(b) Prove that the set consisting of all the cut-sets and the edge-disjoint unions of cut-sets (including the null set ϕ) in a graph G is an abelian group under the ring sum operation.

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M.Sc. DEGREE EXAMINATION, MAY 2011.

Second Year

Mathematics

Paper VI — RINGS AND MODULES

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Show that the class of semilattices can be equationally defined as the class of all semigroups $(s, \wedge)$ satisfying the commutative and idempotent laws : $a \wedge b = b \wedge a$, $a \wedge a = a$.

   (b) Show that in any Boolean algebra $a'' = (a')' = a$.

2. (a) If θ is a reflexive homomorphic relation on a ring, then show that θ is symmetric and transitive, hence a congruence relation.

   (b) If A, B and C are additive subgroups of R then show that $(AB)C = A(BC)$. Moreover $AB \subset C \Leftrightarrow A \subset C : B \Leftrightarrow B \subset A : C$.

3. (a) If $B' \subset B \subset A$ and $C' \subset C \subset A$, then show that $(B' + (B \cap C))/(B' + (B \cap C')) \cong (C' + (B \cap C))/(C' + (B' \cap C))$.

(b) Let B be a submodule of A_R. Then show that A is Artinian if and only if B and A/B are Artinian.

4. (a) If R is subdirectly irreducible and semiprime, then show that R is a field.

(b) Show that every commutative regular ring is semiprimitive.

(c) Show that every ring is a subdirect product of subdirectly irreducible rings.

5. (a) Show that every equivalence class of fractions contains exactly one irreducible fraction, and this extends all fractions in the class.

(b) If R is a commutative ring then show that $Q(R)$ is regular if and only if R is semiprime.

6. (a) Show that the ring R is primitive if and only if there exists a faithful irreducible module A_R.

(b) Show that the prime radical of R is the set of all strongly nilpotent elements.

7. (a) If R is semiprime the show that R_R and R^R have the same socles. More over, they have the same homogeneous components, and these are minimal ideals.

(b) If $e^2 = e \in R$ and $f \in R$, then show that there is a group isomorphism $Hom_R(eR, fR) \cong f\,\mathrm{Re}$. More over, if $f = e$, this is a ring isomorphism.

8. (a) If R is right Aritinian then show that any right R-module is Noetherian if and only if it is Artinian.

(b) Show that in a right Noetherian ring the prime radical is the largest nil left ideal.

9. (a) Show that M_R is free if and only if it is isomorphic to a direct sum of copies of R_R.

(b) Show that every R-module is projective if and only if R is completely reducible.

10. (a) Show that M_R is injective if and only if, for every right ideal K of R and every $\phi \in Hom_R(K, M)$ there exists an $m \in M$ such that $\phi k = mk$ for all $k \in K$.

(b) If R^F is a free module then show that F_R^* is injective.