

GS-2012
(Computer & Systems Sciences)
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **COMPUTER & SYSTEMS SCIENCES - December 11, 2011**

Duration : Three hours (3 hours)

Name : _____ Ref. Code : _____

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.
2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question : this will be treated as a wrong answer.
3. This question paper consists of three (3) parts. **Part-A** contains twenty (20) questions and **must be attempted** by all candidates. **Part-B & Part-C** contain twenty (20) questions each, directed towards candidates for Computer Science and Systems Science, respectively. **ATTEMPT EITHER PART-B OR PART-C BUT NOT BOTH.** All questions carry equal marks. A correct answer for a question will give you +4 marks, a wrong answer will give you -1 mark, and a question not answered will not get you any marks.
4. We advise you to first mark the correct answers in the QUESTION SHEET and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.
5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.
6. **Use of calculators is NOT permitted.**
7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

Part A
Common Questions

1. Amar and Akbar both tell the truth with probability $3/4$ and lie with probability $1/4$. Amar watches a test match and talks to Akbar about the outcome. Akbar, in turn, tells Anthony, "Amar told me that India won". What probability should Anthony assign to India's win?

- (a) $9/16$
- (b) $6/16$
- (c) $7/16$
- (d) $10/16$ ✓
- (e) none of the above

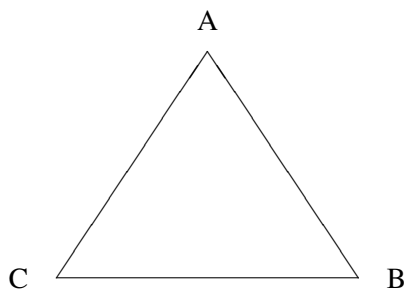
2. If Mr.M is guilty, then no witness is lying unless he is afraid. There is a witness who is afraid. Which of the following statements is true?

(Hint: Formulate the problem using the following predicates

- G – Mr. M is guilty
- $W(x)$ – x is a witness
- $L(x)$ – x is lying
- $A(x)$ – x is afraid)

- (a) Mr.M is guilty
- (b) Mr.M is not guilty
- (c) From these facts one cannot conclude that Mr.M is guilty ✓
- (d) There is a witness who is lying
- (e) No witness is lying.

3. Long ago, in a planet far far away, there lived three races of intelligent inhabitants: the Blues (who always tell the truth), the Whites (who always lie), and the Pinks (who, when asked a series of questions, start with a lie and then tell the truth and lie alternately). To three creatures, chosen from the planet and seated facing each other at A, B and C (see figure), the following three questions are put: (i) What race is your left-hand neighbour? (ii) What race is your right-hand neighbour? (iii) What race are you?



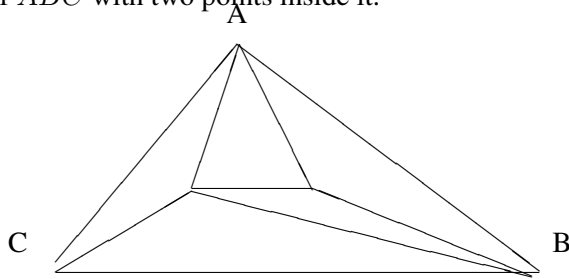
Here are their answers:

A: (i) White (ii) Pink (iii) Blue
B: (i) Pink (ii) Pink (iii) Blue
C: (i) White (ii) Blue (iii) Blue

What is the actual race of each of the three creatures?

- [a] A is Pink, B is White, C is Blue
- [b] A is Blue, B is Pink, C is White
- [c] A is Pink, B is Blue, C is Pink ✓
- [d] A is White, B is pink, C is blue
- [e] Cannot be determined from the above data

4. Let ABC be a triangle with n distinct points inside. A triangulation of ABC with respect to the n points is obtained by connecting as many points as possible such that no more line segment can be added without intersecting other line segments. In other words, ABC has been partitioned into triangles with end points at the n points or at the vertices A, B, C . For example, the following figure gives one possible triangulation of ABC with two points inside it.



Although there are many different ways to triangulate ABC with the n points inside, the **number** of triangles depends only on n . In the above figure it is five. How many triangles are there in a triangulation of ABC with n points inside it?

- (a) $3n - 1$
- (b) $n^2 + 1$
- (c) $n + 3$
- (d) $2n + 1$ ✓
- (e) $4n - 3$

5. What is the maximum number of points of intersection between the diagonals of a convex octagon (8-vertex planar polygon)? Note that a polygon is said to be convex if the line segment joining any two points in its interior lies wholly in the interior of the polygon. Only points of intersection between diagonals that lie in the interior of the octagon are to be considered for this problem.

- (a) 55
- (b) 60
- (c) 65
- (d) 70 ✓
- (e) 75

6. A certain pair of used shoes can be repaired for Rs. 1250 and will last for 1 year. A pair of the same kind of shoes can be purchased new for Rs. 2800 and will last for 2 years. The average cost per year of the new shoes is what percent greater than the cost of repairing the used shoes?

- (a) 5%
- (b) 12% ✓
- (c) 15%
- (d) 3%
- (e) 24%

7. It is required to divide the $2n$ members of a club into n disjoint teams of 2 members each. The teams are not labelled. The number of ways in which this can be done is:

- (a) $\frac{(2n)!}{2^n}$
- (b) $\frac{(2n)!}{n!}$
- (c) $\frac{(2n)!}{2^n \cdot n!}$ ✓
- (d) $n!/2$
- (e) None of the above

8. How many pairs of sets (A, B) are there that satisfy the condition $A, B \subseteq \{1, 2, \dots, 5\}$, $A \cap B = \{\}$?

- (a) 125
- (b) 127
- (c) 130
- (d) 243 ✓
- (e) 257

9. The probability of throwing six perfect dices and getting six different faces is

- (a) $1 - 6!/6^6$
- (b) $6!/6^6$ ✓
- (c) 6^{-6}
- (d) $1 - 6^{-6}$
- (e) None of the above.

10. In how many different ways can r elements be picked from a set of n elements if

- (i) Repetition is not allowed and the order of picking matters?
- (ii) Repetition is allowed and the order of picking does not matter?

- (a) $\frac{n!}{(n-r)!}$ and $\frac{(n+r-1)!}{r!(n-1)!}$, respectively. ✓
- (b) $\frac{n!}{(n-r)!}$ and $\frac{n!}{r!(n-1)!}$, respectively.
- (c) $\frac{n!}{r!(n-r)!}$ and $\frac{(n-r+1)!}{r!(n-1)!}$, respectively.
- (d) $\frac{n!}{r!(n-r)!}$ and $\frac{n!}{(n-r)!}$, respectively.
- (e) $\frac{n!}{r!}$ and $\frac{r!}{n!}$, respectively.

11. Let N be the sum of all numbers from 1 to 1023 except the five primes numbers: 2, 3, 11, 17, 31. Suppose all numbers are represented using two bytes (sixteen bits). What is the value of the least significant byte (the least significant eight bits) of N ?

- (a) 00000000
- (b) 10101110
- (c) 01000000
- (d) 10000000
- (e) 11000000 ✓

12. For the polynomial $p(x) = 8x^{10} - 7x^3 + x - 1$ consider the following statements (which may be true or false)

- (i) It has a root between $[0, 1]$.
- (ii) It has a root between $[0, -1]$.
- (iii) It has no roots outside $(-1, 1)$.

Which of the above statements are true?

- (a) Only (i)
- (b) Only (i) and (ii)
- (c) Only (i) and (iii)
- (d) Only (ii) and (iii)
- (e) All of (i), (ii) and (iii) ✓

13. The maximum value of the function

$$f(x, y, z) = (x - 1/3)^2 + (y - 1/3)^2 + (z - 1/3)^2$$

subject to the constraints

$$x + y + z = 1, \quad x \geq 0, y \geq 0, z \geq 0$$

is

- (a) $1/3$
- (b) $2/3$ ✓
- (c) 1
- (d) $4/3$
- (e) $4/9$

14. The limit $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$ equals

- (a) ∞
- (b) 1
- (c) $1/2$ ✓
- (d) 0
- (e) None of the above

15. Consider the differential equation $dx/dt = (1 - x)(2 - x)(3 - x)$. Which of its equilibria is unstable?

- (a) $x = 0$
- (b) $x = 1$
- (c) $x = 2$ ✓
- (d) $x = 3$
- (e) None of the above

16. Walking at $4/5$ is normal speed a man is 10 minute too late. Find his usual time in minutes.

- (a) 81
- (b) 64
- (c) 52
- (d) 40 ✓
- (e) It is not possible to determine the usual time from given data.

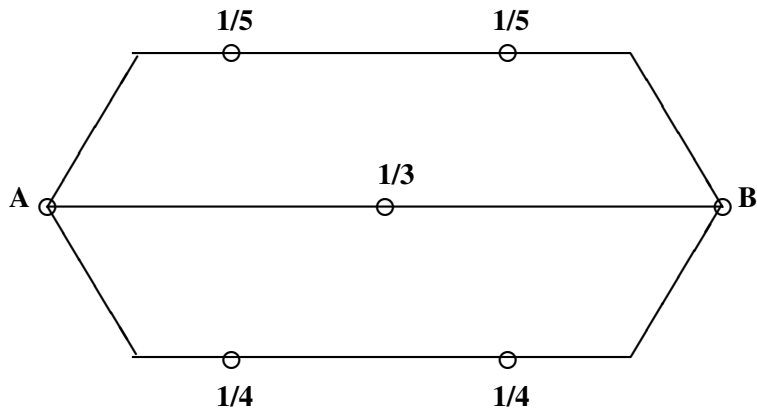
17. A spider is at the bottom of a cliff, and is n inches from the top. Every step it takes brings it one inch closer to the top with probability $1/3$, and one inch away from the top with probability $2/3$, unless it is at the bottom in which case, it always gets one inch closer. What is the expected number of steps for the spider to reach the top as a function of n ?

- (a) It will never reach the top.
- (b) Linear in n
- (c) Polynomial in n
- (d) Exponential in n ✓
- (e) Double exponential in n

18. A large community practices birth control in the following peculiar fashion. Each set of parents continues having children until a son is born; then they stop. What is the ratio of boys to girls in the community if, in the absence of birth control, 51% of the babies are born male?

- (a) 51:49 ✓
- (b) 1:1
- (c) 49:51
- (d) 51:98
- (e) 98:51

19. An electric circuit between two terminals A and B is shown in the figure below, where the numbers indicate the probabilities of failure for the various links, which are all independent.



What is the probability that A and B are connected?

- (a) $\frac{6}{25}$
- (b) $\frac{379}{400}$ ✓
- (c) $\frac{1}{1200}$
- (d) $\frac{1199}{1200}$
- (e) $\frac{59}{60}$

20. There are 1000 balls in a bag, of which 900 are black and 100 are white. I randomly draw 100 balls from the bag. What is the probability that the 101st ball will be black?

- (a) $9/10$ ✓
- (b) More than $9/10$ but less than 1
- (c) Less than $9/10$ but more than 0
- (d) 0
- (e) 1

Part B
Computer Science Questions

1. For $x, y \in \{0, 1\}^n$, let $x \oplus y$ be the element of $\{0, 1\}^n$ obtained by the component-wise exclusive-or of x and y . A Boolean function $F : \{0, 1\}^n \rightarrow \{0, 1\}$ is said to be *linear* if $F(x \oplus y) = F(x) \oplus F(y)$, for all x and y . The number of linear functions from $\{0, 1\}^n$ to $\{0, 1\}$ is

- (a) 2^{2^n}
- (b) 2^{n+1}
- (c) $2^{n-1} + 1$
- (d) $n!$
- (e) 2^n ✓

2. In a graph, the degree of a vertex is the number of edges incident (connected) on it. Which of the following is true for every graph G ?

- (a) There are even number of vertices of even degree
- (b) There are odd number of vertices of even degree
- (c) There are even number of vertices of odd degree ✓
- (d) There are odd number of vertices of odd degree
- (e) All the vertices are of even degree.

3. For a person p , let $w(p)$, $A(p, y)$, $L(p)$ and $J(p)$ denote that p is a woman, p admires y , p is a lawyer and p is a judge respectively. Which of the following is the correct translation in first order logic of the sentence: "All women who are lawyers admire some judge"?

- (a) $\forall x : [(w(x) \wedge L(x)) \Rightarrow (\exists y : (J(y) \wedge w(y) \wedge A(x, y)))]$
- (b) $\forall x : [(w(x) \Rightarrow L(x)) \Rightarrow (\exists y : (J(y) \wedge A(x, y)))]$
- (c) $\forall x \forall y : [(w(x) \wedge L(x)) \Rightarrow (J(y) \wedge A(x, y))]$
- (d) $\exists y \forall x : [(w(x) \wedge L(x)) \Rightarrow (J(y) \wedge A(x, y))]$
- (e) $\forall x : [(w(x) \wedge L(x)) \Rightarrow (\exists y : (J(y) \wedge A(x, y)))]$ ✓

4. Let \wedge, \vee denote the meet and join operations of a lattice. A lattice is called *distributive* if for all x, y, z ,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

It is called *complete* if meet and join exist for every subset. It is called *modular* if for all x, y, z ,

$$z \leq x \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z$$

The positive integers under divisibility ordering i.e. $p \leq q$ if p divides q forms a

- (a) complete lattice
- (b) modular, but not distributive lattice
- (c) distributive lattice ✓
- (d) lattice but not a complete lattice
- (e) under the give ordering positive integers do not form a lattice

5. Let R be a binary relation over a set S . The binary relation R is called an equivalence relation if it is reflexive transitive and symmetric. The relation is called a partial order if it is reflexive, transitive and antisymmetric. (Notation: Let aRb denote that order pair $(a, b) \in R$.) The relation R is called a *well-order* if R is a partial order and there does not exist an infinite descending chain (with respect to R) within S . An infinite sequence x_1, x_2, \dots of elements of S is called an infinite descending chain if for all i we have $x_{i+1}Rx_i$ and $x_i \neq x_{i+1}$.

Take $S = \mathbb{N} \times \mathbb{N}$ and let the binary relation \sqsubseteq over S be such that $(i_1, j_1) \sqsubseteq (i_2, j_2)$ if and only if either $(i_1 < i_2)$ or $((i_1 = i_2) \wedge (j_1 \leq j_2))$. Which statement is true of \sqsubseteq ?

- (a) \sqsubseteq is an equivalence relation but not a well order.
- (b) \sqsubseteq is a partial order but not a well order.
- (c) \sqsubseteq is a partial order and a well order. ✓
- (d) \sqsubseteq is an equivalence relation and a well order.
- (e) \sqsubseteq is neither a partial order nor an equivalence relation.

6. Let n be a large integer. Which of the following statements is TRUE?

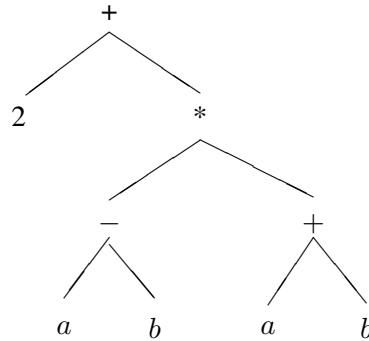
- (a) $2^{\sqrt{2\log n}} < \frac{n}{\log n} < n^{1/3}$
- (b) $\frac{n}{\log n} < n^{1/3} < 2^{\sqrt{2\log n}}$
- (c) $2^{\sqrt{2\log n}} < n^{1/3} < \frac{n}{\log n}$ ✓
- (d) $n^{1/3} < 2^{\sqrt{2\log n}} < \frac{n}{\log n}$
- (e) $\frac{n}{\log n} < 2^{\sqrt{2\log n}} < n^{1/3}$

7. A bag contains 16 balls of the following colors: 8 red, 4 blue, 2 green, 1 black, and 1 white. Anisha picks a ball randomly from the bag, and messages Babu its color using a string of zeros and ones. She replaces the ball in the bag, and repeats this experiment, many times. What is the minimum expected length of the message she has to convey to Babu per experiment?

- (a) 3/2
- (b) $\log 5$

- (c) 15/8 ✓
- (d) 31/16
- (e) 2

8. Consider the parse tree



Assume that * has higher precedence than +, - and operators associate right to left (i.e. $a + b + c = (a + (b + c))$). Consider

- (i) $2 + a - b$
- (ii) $2 + a - b * a + b$
- (iii) $(2 + ((a - b) * (a + b)))$
- (iv) $2 + (a - b) * (a + b)$

The parse tree corresponds to

- (a) Expression (i)
- (b) Expression (ii)
- (c) Expression (iv) only
- (d) Expression (ii), (iii) and (iv)
- (e) Expressions (iii) and (iv) only ✓

9. Consider the concurrent program

```

x:=1;
cobegin
  x := x + x + 1  ||  x := x + 2
coend;

```

Reading and writing of a variable is atomic, but evaluation of an expression is **not** atomic. The set of possible values of variable x at the end of the execution of the program is

- (a) {3}
- (b) {7}
- (c) {3,5,7} ✓
- (d) {3,7}
- (e) {3,5}

10. Consider the *blocked-set semaphore* where the signaling process awakens any one of the suspended processes; i.e.,

Wait(S): If $S > 0$ then $S \leftarrow S - 1$, else suspend the execution of this process.

Signal(S): If there are processes that have been suspended on semaphore S , then wake any one of them, else $S \leftarrow S + 1$.

Consider the following solution of mutual exclusion problem using blocked-set semaphores.

```
S :=1;
cobegin
P(1)  ||  P(2)  ||  ...  ||  P(N)
coend
```

where the task body $P(i)$ is

```
begin
while true do
begin
<non critical section>
Wait(S)
<critical section>
Signal(S)
end
end
```

Here N is the number of concurrent processors. Which of the following is true?

- (a) The program fails to achieve mutual exclusion of critical regions
- (b) The program achieves mutual exclusion; but starvation freedom is ensured only for $N \leq 2$ ✓
- (c) The program does not ensure mutual exclusion if $N \geq 3$
- (d) The program achieves mutual exclusion, but allows starvation for any $N \geq 2$
- (e) The program achieves mutual exclusion and starvation freedom for any $N \geq 1$.

11. Consider the following three versions of the binary search program. Assume that the elements of type T can be compared with each other; also assume that the array a is sorted.

```
i, j, k : integer;
a : array [1..N] of T;
x : T;
```

```
Program 1 :   i := 1; j := N;
              repeat
                k := (i+j) div 2;
                if a[k] < x then i := k else j := k
              until (a[k] = x) or (i > j)
```

```
Program 2 :   i := 1; j := N;
              repeat
                k := (i+j) div 2;
                if x < a [k] then j := k-1;
                if a[k] < x then i := k+1;
              until i > j
```

```
Program 3 :   i := 1; j := N;
              repeat
                k := (i+j) div 2;
                if x < a[k] then j := k else i := k+1
              until i > j
```

A binary search program is called correct provided it terminates with $a[k] = x$ whenever such an element exists, or it terminates with $a[k] \neq x$ if there exists no array element with value x . Which of the following statements is correct?

- (a) Only Program 1 is correct
- (b) Only Program 2 is correct
- (c) Only Programs 1 and 2 are correct
- (d) Both Programs 2 and 3 are correct
- (e) All the three programs are wrong ✓

12. Let A be a matrix such that $A^k = 0$. What is the inverse of $I - A$?

- (a) 0
- (b) I
- (c) A
- (d) $1 + A + A^2 + \dots + A^{k-1}$ ✓

(e) Inverse is not guaranteed to exist.

13. An array A contains n integers. We wish to sort A in ascending order. We are told that initially no element of A is more than a distance k away from its final position in the sorted list. Assume that n and k are large and k is much smaller than n . Which of the following is true for the worst case complexity of sorting A ?

- (a) A can be sorted with constant $\cdot kn$ comparisons but not with fewer comparisons
- (b) A cannot be sorted with less than constant $\cdot n \log n$ comparisons
- (c) A can be sorted with constant $\cdot n$ comparisons
- (d) A can be sorted with constant $\cdot n \log k$ comparisons but not with fewer comparisons ✓
- (e) A can be sorted with constant $\cdot k^2n$ comparisons but not fewer

14. Consider the quicksort algorithm on a set of n numbers, where in every recursive subroutine of the algorithm, the algorithm chooses the median of that set as the pivot. Then which of the following statements is TRUE?

- (a) The running time of the algorithm is $\Theta(n)$.
- (b) The running time of the algorithm is $\Theta(n \log n)$. ✓
- (c) The running time of the algorithm is $\Theta(n^{1.5})$.
- (d) The running time of the algorithm is $\Theta(n^2)$.
- (e) None of the above.

15. Let T be a tree on n nodes. Consider the following algorithm, that constructs a sequence of leaves u_1, u_2, \dots . Let u_1 be some leaf of tree. Let u_2 be a leaf that is farthest from u_1 . Let u_3 be the leaf that is farthest from u_2 , and, in general, let u_{i+1} be a leaf of T that is farthest from u_i (if there are many choices for u_{i+1} , pick one arbitrarily). The algorithm stops when some u_i is visited again. What can you say about the distance between u_i and u_{i+1} , as $i = 1, 2, \dots$?

- (a) For some trees, the distance strictly reduces in each step.
- (b) For some trees, the distance increases initially and then decreases.
- (c) For all trees, the path connecting u_2 and u_3 is a longest path in the tree. ✓
- (d) For some trees, the distance reduces initially, but then stays constant.
- (e) For the same tree, the distance between the last two vertices visited can be different, based on the choice of the first leaf u_1 .

16. Consider a complete binary tree of height n , where each edge is a one Ohm resistor. Suppose all the leaves of the tree are tied together. Approximately how much is the effective resistance from the root to this bunch of leaves for very large n ?

- (a) Exponential in n

- (b) Cubic in n
- (c) Linear in n
- (d) Logarithmic in n
- (e) Of the order square root of n



Question No. 16 has been marked correct for all candidates who have attempted this section and have been given +4 marks.

17. Which of the following correctly describes $LR(k)$ parsing?

- (a) The input string is alternately scanned left to right and right to left with k reversals.
- (b) Input string is scanned once Left to Right with rightmost derivation and k symbol look-ahead. ✓
- (c) $LR(k)$ grammars are expressively as powerful as context-free grammars.
- (d) Parser makes k left-to-right passes over input string.
- (e) Input string is scanned from left to right once with k symbol to the right as look-ahead to give left-most derivation.

18. Let a^i denote a sequence $a \cdot a \cdot \dots \cdot a$ with i letters and let \mathbb{N} be the set of natural numbers $\{1, 2, \dots\}$. Let $L_1 = \{a^i b^{2i} \mid i \in \mathbb{N}\}$ and $L_2 = \{a^i b^{i^2} \mid i \in \mathbb{N}\}$ be two languages. Which of the following is correct?

- (a) Both L_1 and L_2 are context free languages.
- (b) L_1 is context-free and L_2 is recursive but not context-free. ✓
- (c) Both L_1 and L_2 are recursive but not context-free.
- (d) L_1 is regular and L_2 is context-free.
- (e) Complement of L_2 is context-free.

19. Which of the following statements is TRUE?

- (a) Every Turing machine recognizable language is recursive.
- (b) The complement of every recursively enumerable language is recursively enumerable.
- (c) The complement of a recursive language is recursively enumerable. ✓
- (d) The complement of a context free language is context free.
- (e) The set of turing machines which do not halt on empty input forms a recursively enumerable set.

20. This question concerns the classes P and NP . If you are familiar with them, you may skip the definitions and go directly to the question.

Let L be a set. We say that L is in P if there is some algorithm which given input x decides if x is in L or not in time bounded by a polynomial in the length of x . For example, the set of all connected graphs is in P , because there is an algorithm which, given a graph $graph$, can decide if it is connected or not in time roughly proportional to the number of edges of the graph.

The class NP is a superset of class P . It contains those sets that have membership witnesses that can be verified in polynomial time. For example, the set of composite numbers is in NP . To see this take the

witness for a composite number to be one of its divisors. Then the verification process consists of performing just one division using two reasonable size numbers. Similarly, the set of those graphs that have a Hamilton cycle, i.e. a cycle containing all the vertices of the graph, is in NP. To verify that the graph has a Hamilton cycle we just check if the witnessing sequence of vertices indeed a cycle of the graph that passes through all the vertices of the graph. This can be done in time that is polynomial in the size of the graph.

More precisely, if L is a set in P consisting of elements of the form (x, w) , then the set

$$M = \{x : \exists w, |w| \leq |x|^k \text{ and } (x, w) \in L\},$$

is in NP.

Let $G = (V, E)$ be a graph. G is said to have perfect matching if there is a subset M of the edges of G so that

- (i) No two edges in M intersect (have a vertex in common); and
- (ii) Every vertex of G has an edge in M .

Let MATCH be the set of all graphs that have a perfect matching. Let $\overline{\text{MATCH}}$ be the set of graphs that do not have a perfect matching. Let $o(G)$ be the number of components of G that have an odd number of vertices.

Tutte's Theorem: $G \in \text{MATCH}$ if and only if for all subsets S of V , the number of components in $G - S$ (the graph formed by deleting the vertices in S) with an odd number of vertices is at most $|S|$. That is,

$$G \in \text{MATCH} \leftrightarrow \forall S \subseteq V \ o(G - S) \leq |S|.$$

Which of the following is true?

- (a) $\text{MATCH} \in NP$ and $\overline{\text{MATCH}} \notin NP$
- (b) $\overline{\text{MATCH}} \in NP$ and $\text{MATCH} \notin NP$
- (c) $\text{MATCH} \in NP$ and $\overline{\text{MATCH}} \in NP$ ✓
- (d) $\text{MATCH} \notin P$ and $\overline{\text{MATCH}} \notin P$
- (e) none of the above

Part C
Systems Science Questions

1. The minimum value of $f(x) = \ln(1 + \exp(x^2 - 3x + 2))$ for $x \geq 0$, where $\ln(\cdot)$ denotes the natural logarithm, is

- (a) $\ln(1 + e^{-1/4})$ ✓
- (b) $\ln(5/3)$
- (c) 0
- (d) $\ln(1 + e^2)$
- (e) None of the above

← Question No. 1 has been marked correct for all candidates who have attempted this section and have been given +4 marks.

2. Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be complex numbers. Then

$$\lim_{n \rightarrow \infty} \left| \sum_{i=1}^k \alpha_i^n \right|^{1/n}$$

is

- (a) 0
- (b) ∞
- (c) α_k
- (d) α_1
- (e) $\max_j |\alpha_j|$ ✓

3. A sequence of numbers $(x_n : n = 1, 2, 3, \dots)$ is said to have a limit x , if given any number $\epsilon > 0$, there exists an integer n_ϵ such that

$$|x_n - x| < \epsilon$$

for all $n \geq n_\epsilon$. In other words a sequence $(x_n : n = 1, 2, 3, \dots)$ has a limit x all but a finite number of points of this sequence are arbitrarily close to x . Now consider a sequence

$$x_n = 5 + \frac{(-1)^n}{n} + \left(1 - \frac{271}{2^n}\right)$$

for all $n \geq 1$. Which of the following statements is true?

- (a) The sequence $(x_n : n = 1, 2, 3, \dots)$ fluctuates around 6 and has a limit that equals 6. ✓
- (b) The sequence $(x_n : n = 1, 2, 3, \dots)$ oscillates around 5 and does not have a limit.
- (c) The sequence $(x_n : n = 1, 2, 3, \dots)$ oscillates around 6 and does not have a limit.
- (d) The sequence $(x_n : n = 1, 2, 3, \dots)$ is eventually always below 6 and has a limit that equals 6.

- (e) None of the above statements are true.
4. The signal $x_n = 0$ for $n < 0$ and $x_n = a^n/n!$ for $n \geq 0$. Its z-transform $X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$ is
- (a) $1/(z^{-1} - a)$, region of convergence (ROC): $|z| \leq 1/a$
 - (b) $1/(1 - az^{-1})$, ROC: $|z| \geq a$
 - (c) $1/(1 - az^{-1})$, ROC: $|z| > a$
 - (d) Item (a) if $a > 1$, Item (b) if $a < 1$
 - (e) $\exp(az^{-1})$, ROC: entire complex plane ✓
5. Consider a periodic square wave $f(t)$ with a period of 1 second such that $f(t) = 1$ for $t \in [0, 1/2)$ and $f(t) = -1$ for $t \in [1/2, 1)$. It is passed through an ideal low-pass filter with cutoff at 2 Hz. Then the output is
- (a) $\sin(2\pi t)$ ✓
 - (b) $\cos(2\pi t)$
 - (c) $\sin(2\pi t) - \sin(6\pi t)/3 + \sin(10\pi t)/5 - \dots$
 - (d) $\sin(2\pi t) - \cos(2\pi t)$
 - (e) None of the above
6. Let $u(t)$ be the unit step function that takes value 1 for $t \geq 0$ and is zero otherwise. Let $f(t) = e^{-t}u(t)$ and $g(t) = u(t)u(1 - t)$. Then the convolution of $f(t)$ and $g(t)$ is
- (a) $(e - 1)e^{-t}u(t)$
 - (b) $1 - e^{-t}$ for $0 \leq t \leq 1$, $(e - 1)e^{-t}$ for $t \geq 1$ and zero otherwise ✓
 - (c) $te^{-t}u(t)$
 - (d) The convolution integral is not well defined
 - (e) None of the above
7. A linear time-invariant system has a transfer function $H(s) = 1/(1 + s)$. If the input to the system is $\cos(t)$, the output is
- (a) $(e^{jt} + e^{-jt})/2$ where $j = \sqrt{-1}$
 - (b) $\cos(t)/2$
 - (c) $(\cos(t) + \sin(t))/2$ ✓
 - (d) $\sin(t)/2$.
 - (e) The system is unstable and the output is not well-defined.

8. The input to a series RLC circuit is a sinusoidal voltage source and the output is the current in the circuit. Which of the following is true about the magnitude frequency response of this system?

- (a) Depending on the values of R , L and C , a steady state may not exist, and the magnitude frequency response is not well-defined.
- (b) It is low-pass with 3 dB bandwidth $1/(2\pi\sqrt{LC})$.
- (c) It is high-pass with 3 dB bandwidth $1/(2\pi\sqrt{LC})$.
- (d) It is low-pass and the 3-dB bandwidth depends on all: R , L , C .
- (e) It is 0 at DC, decays to zero as frequency increases to infinity, and has a maximum at $1/(2\pi\sqrt{LC})$. ✓

9. $x(t)$ is a signal of bandwidth 4 kHz. It was sampled at a rate of 16 kHz.

$$x_n = x(nT), \quad n \text{ integer}, \quad T = \frac{1}{16} \text{ ms.}$$

Due to a data handling error alternate samples were erased and set to 0.

$$y_n = \begin{cases} x_n, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

Without realizing this error an engineer uses sinc interpolation to try to reconstruct $x(t)$. She obtains

$$y(t) = \sum_{n=-\infty}^{\infty} y_n \operatorname{sinc}\left(\frac{t - nT}{T}\right), \quad T = \frac{1}{16} \text{ ms.}$$

Using which of the procedures below can she recover $cx(t)$ from $y(t)$ where c is some non-zero scaling factor?

- (a) resample $y(t)$ at 16 kHz and sinc interpolate using $T = \frac{1}{8}$ ms
- (b) resample $y(t)$ at 8 kHz and sinc interpolate using $T = \frac{1}{16}$ ms
- (c) send $y(t)$ over a low pass filter of bandwidth 4 kHz ✓
- (d) any of the above
- (e) none of the above

10. Suppose three dice are rolled independently. Each dice can take values 1 to 6 with equal probability. Find the probability that the second highest outcome equals the average of the other two outcomes. Here, the ties may be resolved arbitrarily.

- (a) 1/6
- (b) 1/9
- (c) 39/216

(d) $7/36$ ✓

(e) $43/216$

11. A Poisson random variable X is given by $\Pr\{X = k\} = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \dots$ for $\lambda > 0$. The variance of X scales as

(a) λ ✓

(b) λ^2

(c) λ^3

(d) $\sqrt{\lambda}$

(e) None of the above

12. In modeling the number of health insurance claims filed by an individual during a three year period, an analyst makes a simplifying assumption that for all non-negative integer up to 5,

$$p_{n+1} = \frac{1}{2}p_n$$

where p_n denotes the probability that a health insurance policy holder files n claims during this period. The analyst assumes that no individual files more than 5 claims in this period. Under these assumptions, what is the probability that a policy holder files more than two claims in this period?

(a) $7/31$

(b) $29/125$

(c) $1/3$

(d) $13/125$

(e) None of the above ✓

13. Consider a single amoeba that at each time slot splits into two with probability p or dies otherwise with probability $1 - p$. This process is repeated independently infinitely at each time slot, i.e. if there are any amoebas left at time slot t , then they all split independently into two amoebas with probability p or die with probability $1 - p$. Which of the following is the expression for the probability that the race of amoeba becomes extinct.

(a) $\min\left\{\frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p}\right\}$ ✓

(b) $\min\left\{\frac{1 \pm \sqrt{1 + 4p}}{2p}\right\}$

(c) $\min\left\{\frac{-1 \pm \sqrt{1 + 4p(1-p)}}{2p}\right\}$

(d) $\min\left\{\frac{1 \pm \sqrt{1 - 4p(1-p)}}{2(1-p)}\right\}$

(e) None of the above

14. Let X and Y be independent, identically distributed standard normal random variables, i.e., the probability density function of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad -\infty < x < \infty.$$

The random variable Z is defined as $Z = aX + bY$, where a and b are non-zero real numbers. What is the probability density function of Z ?

- (a) $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$
- (b) $\frac{1}{\sqrt{2\pi(a^2+b^2)}} \exp\left(-\frac{z^2}{2(a^2+b^2)}\right)$ ✓
- (c) $\frac{|z|}{2} \exp\left(-\frac{z^2}{2}\right)$
- (d) $\frac{|z|}{2(a^2+b^2)} \exp\left(-\frac{z^2}{2(a^2+b^2)}\right)$
- (e) none of the above

15. Consider a string of length 1 m. Two points are chosen independently and uniformly random on it thereby dividing the string into three parts. What is the probability that the three parts can form the sides of a triangle?

- (a) 1/4 ✓
- (b) 1/3
- (c) 1/2
- (d) 2/3
- (e) 3/4

16. Let P be a $n \times n$ matrix such that $P^k = \mathbf{0}$, for some $k \in \mathbb{N}$ and where $\mathbf{0}$ is an all zeros matrix. Then **at least** how many eigenvalues of P are zero

- (a) 1
- (b) $n - 1$
- (c) n ✓
- (d) 0
- (e) None of the above

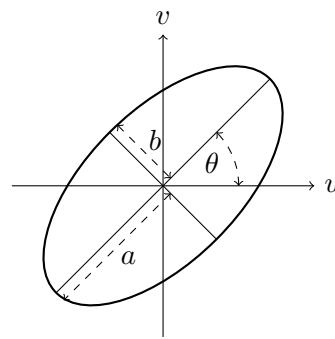
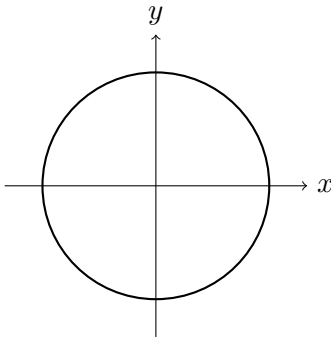
17. Let $A = U\Lambda U^\dagger$ be a $n \times n$ matrix, where $UU^\dagger = I$. Which of the following statements is TRUE.

- (a) The matrix $I + A$ has non-negative eigen values

- (b) The matrix $I + A$ is symmetric
- (c) $\det(I + A) = \det(I + \Lambda)$
- (d) (a) and (c)
- (e) (b) and (c)
- (f) (a), (b) and (c)

← Question No. 17 has been marked correct for all candidates who have attempted this section and have been given +4 marks.

18. Under a certain coordinate transformation from (x, y) to (u, v) the circle $x^2 + y^2 = 1$ shown below on the left side was transformed into the ellipse shown on the right side.



If the transformation is of the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix},$$

which of the following could the matrix \mathbf{A} be:

$$A_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$A_3 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- (a) A_1 only
- (b) A_2 only
- (c) A_1 or A_2
- (d) A_1 or A_3
- (e) A_2 or A_3 ✓

19. X and Y are two 3 by 3 matrices. If

$$XY = \begin{pmatrix} 1 & 3 & -2 \\ -4 & 2 & 5 \\ 2 & -8 & -1 \end{pmatrix},$$

then

- (a) X has rank 2
- (b) at least one of X, Y is not invertible ✓
- (c) X can't be an invertible matrix
- (d) X and Y could both be invertible.
- (e) None of the above

20. Let A be a 2×2 matrix with all entries equal to 1. Define $B = \sum_{n=0}^{\infty} A^n/n!$. Then

(a) $B = e^2 A/2$

(b)

$$B = \begin{pmatrix} 1+e & e \\ e & 1+e \end{pmatrix}$$

(c)

$$B = \frac{1}{2} \begin{pmatrix} e^2+1 & e^2-1 \\ e^2-1 & e^2+1 \end{pmatrix} \checkmark$$

(d)

$$B = \begin{pmatrix} 1+e^2 & e^2 \\ e^2 & 1+e^2 \end{pmatrix}$$

(e) None of the above

GS-2012 (COMPUTER & SYSTEMS SCIENCES)

ANSWER SHEET

Please see reverse for instructions on filling of answer sheet.

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Ref Code		1	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
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Signature of the Student

INSTRUCTIONS

The Answer Sheet is machine-readable. Apart from filling in the details asked for in the answer sheet, please make sure that the Reference Code is filled by blackening the appropriate circles in the box provided on the right-top corner. Only use HB pencils to fill-in the answer sheet.

e.g. if your reference code is 15207 :

Reference Code :					
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Also, the multiple choice questions are to be answered by blackening the appropriate circles as described below

e.g. if your answer to question 1 is (b)
and your answer to question 2 is (d)
then

PART-A					
	(a)	(b)	(c)	(d)	(e)
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