MTH

GS-2013 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 9, 2012**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark; each wrong answer will get a -1 mark, and a question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

3. There are forty (40) questions divided into four parts, Part-A, Part-B, Part-C and Part-D. Each Part consists of 10 True-False questions.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

8. See the back of this page for Notation and Conventions used in this test.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers

$\mathbb{Z}$:= Set of integers

$\mathbb{Q}$:= Set of rational numbers

$\mathbb{R}$:= Set of real numbers

$\mathbb{C}$:= Set of complex numbers

$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$, the open interval

A sequence is always indexd by natural numbers.

Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology.

INSTRUCTIONS

THERE ARE 4 PARTS AND 40 QUESTIONS IN TOTAL, CONSISTING OF 10 QUESTIONS IN EACH PART.

EVERY CORRECT ANSWER CARRIES +1 MARK AND EVERY WRONG ANSWER CARRIES −1 MARK.

PART A

F 1. Every countable group G has only countably many distinct subgroups.

T 2. Any automorphism of the group $\mathbb{Q}$ under addition is of the form $x \mapsto qx$ for some $q \in \mathbb{Q}$.

T 3. The equation $x^3 + 3x - 4 = 0$ has exactly one real root.

T 4. The equation $x^3 + 10x^2 - 100x + 1729 = 0$ has at least one complex root α such that $|\alpha| > 12$.

F 5. All non-trivial proper subgroups of $(\mathbb{R}, +)$ are cyclic.

F 6. Every infinite abelian group has at least one element of infinite order.

F 7. If A and B are similar matrices then every eigenvector of A is an eigenvector of B.

T 8. If a real square matrix A is similar to a diagonal matrix and satisfies $A^n = 0$ for some n, then A must be the zero matrix.

T 9. There is an element of order 51 in the multiplicative group $(\mathbb{Z}/103\mathbb{Z})^*$.

T 10. Any normal subgroup of order 2 is contained in the center of the group.

PART B

F 11. Consider the sequences

$$x_n = \sum_{j=1}^{n} \frac{1}{j}$$
$$y_n = \sum_{j=1}^{n} \frac{1}{j^2}$$

Then $\{x_n\}$ is Cauchy but $\{y_n\}$ is not.

F 12. $\lim_{x\to 0} \frac{\sin(x^2)}{x^2} \sin\left(\frac{1}{x}\right) = 1$.

F 13. Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ be Riemann integrable functions defined on the closed intervals $[a, b]$ and $[c, d]$ respectively. Then the composite $g \circ f$ is also Riemann integrable.

T 14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x^3$. Then f is continuous but not uniformly continuous.

T 15. Let $x_1 \in (0, 1)$ be a real number between 0 and 1. For $n > 1$, define

$$x_{n+1} = x_n - x_n^{n+1}.$$

Then $\lim_{n\to\infty} x_n$ exists.

T 16. Suppose $\{a_i\}$ is a sequence in $\mathbb{R}$ such that $\sum |a_i||x_i| < \infty$ whenever $\sum |x_i| < \infty$. Then $\{a_i\}$ is a bounded sequence.

T 17. The integral $\int_0^\infty e^{-x^5} dx$ is convergent.

F 18. Let $P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ where n is a large positive integer. Then $\lim_{x\to\infty} \frac{e^x}{P(x)} = 1$.

F 19. Every differentiable function $f : (0, 1) \to [0, 1]$ is uniformly continuous.

T 20. Consider the function $f(x) = ax + b$ with $a, b \in \mathbb{R}$. Then the iteration

$$x_{n+1} = f(x_n); \qquad n \geq 0$$

for a given x_0 converges to $b/(1 - a)$ whenever $0 < a < 1$.

PART C

F 21. Every homeomorphism of the 2-sphere to itself has a fixed point.

F 22. The intervals $[0, 1)$ and $(0, 1)$ are homeomorphic.

F 23. Let X be a complete metric space such that distance between any two points is less than 1. Then X is compact.

F 24. There exists a continuous surjective function from S^1 onto $\mathbb{R}$.

T 25. There exists a complete metric on the open interval $(0, 1)$ inducing the usual topology.

F 26. There exists a continuous surjective map from the complex plane onto the non-zero reals.

T 27. If every differentiable function on a subset $X \subset \mathbb{R}^n$ (i.e., restriction of a differentiable function on a neighbourhood of X) is bounded, then X is compact.

F 28. Let $f : X \to Y$ be a continuous map between metric spaces. If f is a bijection, then its inverse is also continuous.

T 29. Let f be a function on the closed interval $[0, 1]$ defined by

$$f(x) = x \quad \text{if } x \text{ is rational}$$

$$f(x) = x^2 \quad \text{if } x \text{ is irrational}$$

Then f is continuous at 0 and 1.

T 30. There exists an infinite subset $S \subset \mathbb{R}^3$ such that any three vectors in S are linearly independent.

PART D

F 31. The inequality

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n}}$$

is false for all n such that $101 \le n \le 2000$.

F 32. $\lim_{n\to\infty} (n+1)^{1/3} - n^{1/3} = \infty$.

T 33. There exists a bijection between $\mathbb{R}^2$ and the open interval $(0,1)$.

F 34. Let S be the set of all sequences $\{a_1, a_2, ..., a_n, ...\}$ where each entry a_i is either 0 or 1. Then S is countable.

T 35. Let $\{a_n\}$ be any non-constant sequence in $\mathbb{R}$ such that $a_{n+1} = \frac{a_n + a_{n+2}}{2}$ for all $n \ge 1$. Then $\{a_n\}$ is unbounded.

F 36. The function $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(n) = n^3 - 3n$ is injective.

F 37. The polynomial $x^3 + 3x - 2\pi$ is irreducible over $\mathbb{R}$.

T 38. Let V be the vector space consisting of polynomials with real coefficients in variable t of degree ≤ 9. Let $D : V \to V$ be the linear operator defined by

$$D(f) := \frac{df}{dt}.$$

Then 0 is an eigenvalue of D.

T 39. If A is a complex $n \times n$ matrix with $A^2 = A$, then rank $A =$ trace A.

F 40. The series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

is divergent.