Sr.No. 22791

E

SUBJECT CODE: **4** BOOKLET CODE: **C**

2011 (II)
MATHEMATICAL SCIENCES
TEST BOOKLET

Time : 3:00 Hours *Maximum Marks: 200*

INSTRUCTIONS

1. **You have opted for English as medium of Question Paper.** This Test Booklet contains one hundred and twenty (20 Part'A'+40 Part 'B' +60 Part 'C') Multiple Choice Questions (MCQs). You are required to answer a maximum of 15, 25 and 20 questions from part 'A' 'B' and 'C' respectively. If more than required number of questions are answered, only first 15, 25 and 20 questions in Parts 'A' 'B' and 'C' respectively, will be taken up for evaluation.
2. Answer sheet has been provided separately. Before you start filling up your particulars, please ensure that the booklet contains requisite number of pages and that these are not torn or mutilated. If it is so, you may request the Invigilator to change the booklet. Likewise, check the answer sheet also. Sheets for rough work have been appended to the test booklet.
3. Write your Roll No., Name, Your address and Serial Number of this Test Booklet on the Answer sheet in the space provided on the side 1 of Answer sheet. Also put your signatures in the space identified.
4. **You must darken the appropriate circles with a pencil related to Roll Number, Subject Code, Booklet Code and Centre Code on the OMR answer sheet. It is the sole responsibility of the candidate to meticulously follow the instructions given on the Answer Sheet, failing which, the computer shall not be able to decipher the correct details which may ultimately result in loss, including rejection of the OMR answer sheet.**
5. Each question in Part 'A' carries **2** marks, Part 'B' **3** marks and Part 'C' **4.75** marks respectively. There will be negative marking @ 0.5marks in Part 'A' and @0.75 in Part 'B' for each wrong answer and no negative marking for Part 'C'.
6. Below each question in Part 'A' and 'B', four alternatives or responses are given. Only one of these alternatives is the "correct" option to the question. You have to find, for each question, the correct or the best answer. In Part 'C' each question may have **'ONE'** or **'MORE'** correct options. Credit in a question shall be given only on identification of **'ALL'** the correct options in Part 'C'. No credit shall be allowed in a question if any incorrect option is marked as correct answer.
7. Candidates found copying or resorting to any unfair means are liable to be disqualified from this and future examinations.
8. Candidate should not write anything anywhere except on answer sheet or sheets for rough work.
9. After the test is over, you MUST hand over the answer sheet (OMR) to the invigilator.
10. Use of calculator is not permitted.

Roll No.......................

Name ....

I have verified all the information filled in by the candidate.

....
Signature of the Invigilator

PART A

1. During a total solar eclipse occurring at noon, it becomes dark enough for a few minutes for stars to become visible. The stars that are seen are those which will be seen from the same location

1. on the following night only
2. on the night one month later
3. on the night three months later
4. on the night six months later

2. A cupboard is filled with a large number of balls of 6 different colours. You already have one ball of each colour. If you are blind-folded, how many balls do you need to draw to be sure of having 3 colour-matched pairs of balls?

(1) 3 (2) 4
(3) 5 (4) 6

3. Restriction endonuclease cleaves DNA molecules at specific 'recognition sites'. One such enzyme has four recognition sites on a circular DNA molecule. After complete digestion, how many fragments would be produced upon reaction with this enzyme?

(1) 4 (2) 5
(3) 3 (4) 6

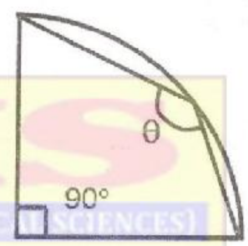
4. In Δ ABC, angle A is larger than angle C and smaller than angle B by the same amount. If angle B is 67°, angle C is

(1) 67° (2) 53°
(3) 60° (4) 57°

5. Living beings get energy from food through the process of aerobic respiration. One of the reactants is

(1) carbon dioxide (2) water vapour
(3) oxygen (4) phosphorus

6. What is the angle θ in the quadrant of a circle shown below?



1. 135°
2. 90°
3. 120°
4. May have any value between 90° and 120°

7. On exposure to desiccation, which of the following bacteria are least likely to experience rapid water loss?

1. Isolated rods
2. Rods in chain
3. Cocci in chain
4. Cocci in clusters

8. See the following mathematical manipulations.

(i) Let $x = 5$
(ii) then $x^2 - 25 = x - 5$
(iii) $(x - 5)(x + 5) = x - 5$
(iv) $x + 5 = 1$ [cancelling $(x - 5)$ from both sides]
(v) $10 = 1$ [Putting $x = 5$]

Which of the above is the wrong step?

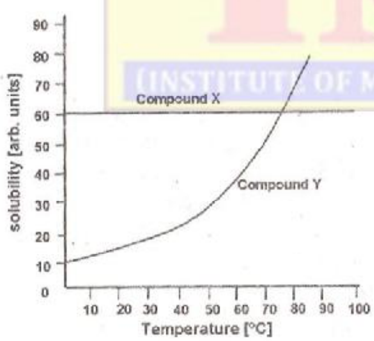
(1) (i) to (ii) (2) (ii) to (iii)

(3) (iii) to (iv) (4) (iv) to (v)

9. Inner planets of the solar system are rocky, whereas outer planets are gaseous. One of the reasons for this is that

1. solar heat drove away the gases to the outer region of the solar system
2. gravitational pull of the sun pulled all rocky material to the inner solar system
3. outer planets are larger than the inner planets
4. comets delivered the gaseous materials to the outer planets

10. The variation of solubilities of two compounds X and Y in water with temperature is depicted below. Which of the following statements is true?



1. Solubility of Y is less than that of X
2. Solubility of X varies with temperature
3. Solubilities of X and Y are the same at 75°C
4. Solubilities of X and Y are independent of temperature

11. The number of craters observed due to meteoritic impacts during the early stages of formation, is less on the Earth than that on the Moon because,

1. formation of craters on the Earth was difficult due to the presence of hard rocks
2. impacting bodies on the Earth were smaller in size
3. craters on the Earth are now covered by ocean water
4. earlier craters are not preserved due to continuous modification of Earth's sŭrface by geological processes

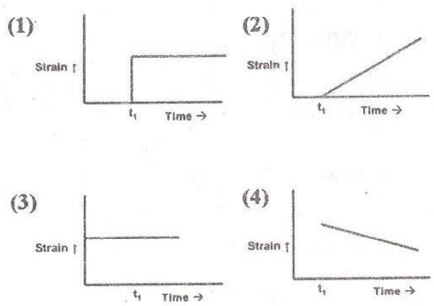
12. Which of the following statements about the concentration of CO_2 in the Earth's atmosphere is true?

1. It was the highest in the very early atmosphere of the Earth
2. It has steadily decreased since the formation of the Earth's atmosphere
3. It has steadily increased since the formation of the Earth's atmosphere
4. Its levels today are the highest in the Earth's history

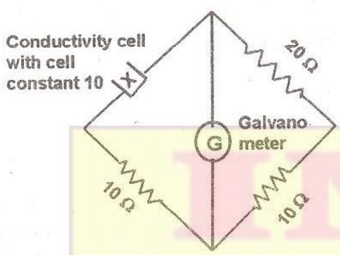
13. When a magnet is made to fall free in air, it falls with an acceleration of 9.8 m s^{-2}. But when it is made to fall through a long aluminium cylinder, its acceleration decreases, because

1. a part of the gravitational potential energy is lost in heating the magnet
2. a part of the gravitational potential energy is lost in heating the cylinder
3. the said experiment was done in the magnetic northern hemisphere
4. the cylinder shields the gravitational force

14. For an elastic material, strain is proportional to stress. A constant stress is applied at time t_1. sWhich of the following plots characterizes the strain in that material?



15. The conductance of a potassium chloride solution is measured using the arrangement depicted below. The specific conductivity of the solution in Sm^{-1}, when there is no deflection in the galvanometer, is



(1) 1.0 (2) 0.5

(3) 2.0 (4) 1.5

16. Magnesium powder, placed in an air-tight glass container at 1.0 bar, is burnt by focusing sunlight. Part of the magnesium burns off, and some is left behind. The pressure of the air in the container after it has returned to room temperature is approximately

(1) 1.0 bar (2) 0.2 bar

(3) 1.2 bar (4) 0.8 bar

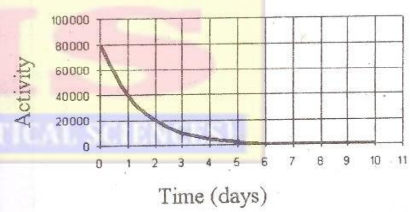
17. A bell is rung before giving food to a dog. After doing this continuously for 10 days, which of the following is most likely to happen?

1. The dog learns to ignore the bell
2. The dog salivates on hearing the bell
3. The dog ignores food and runs towards the bell
4. The dog will not eat food without hearing the bell

18. An overweight person runs 4 km everyday as an exercise. After losing 20% of his body weight, if he has to run the same distance in the same time, the energy expenditure would be

1. 20% more
2. the same as earlier
3. 20% less
4. 40% less

19. What is the half-life of the radio isotope whose activity profile is shown below?



(1) 1 day (2) 3 days

(3) 2 days (4) 4 days

20. A solid cube of side L floats on water with 20% of its volume under water. Cubes identical to it are piled one-by-one on it. Assume that the cubes do not slip or topple, and the contact between their surfaces is perfect. How many cubes are required to submerge one cube completely?

(1) 4 (2) 5

(3) 6 (4) Infinite

PART B

21. The determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ is

(1) 0 (2) -9

(3) -27 (4) 1

22. Which of the following statements is true?

1. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim_{x\to\infty} \frac{\log x}{x} = \infty$

2. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = \infty$ and $\lim_{x\to\infty} \frac{\log x}{x} = 0$

3. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim_{x\to\infty} \frac{\log x}{x} = 0$

4. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ but $\lim_{x\to\infty} \frac{\log x}{x}$ does not exist

23. For a positive integer n, let P_n denote the space of all polynomials $p(x)$ with coefficients in $\mathbb{R}$ such that $\deg p(x) \le n$, and let B_n denote the standard basis of P_n given by $B_n = \{1, x, x^2, \ldots, x^n\}$. If $T : P_3 \to P_4$ is the linear transformation defined by $T(p(x)) = x^2 p'(x) + \int_0^x p(t)\,dt$ and $A = (a_{ij})$ is the 5×4 matrix of T with respect to standard bases B_3 and B_4, then

1. $a_{32} = \frac{3}{2}$ and $a_{33} = \frac{7}{3}$

2. $a_{32} = \frac{3}{2}$ and $a_{33} = 0$

3. $a_{32} = 0$ and $a_{33} = \frac{7}{3}$

4. $a_{32} = 0$ and $a_{33} = 0$

24. Consider the power series $\sum_{n\ge1} a_n z^n$ where a_n = number of divisors of n^{50}. Then the radius of convergence of $\sum_{n\ge1} a_n z^n$ is

(1) 1 (2) 50

(3) $\frac{1}{50}$ (4) 0

25. Let p be a prime number. The order of a p-Sylow subgroup of the group $GL_{50}(\mathbb{F}_p)$ of invertible 50×50 matrices with entries from the finite field $\mathbb{F}_p$, equals:

(1) p^{50} (2) p^{125}

(3) p^{1250} (4) p^{1225}

26. Let X be a connected subset of real numbers. If every element of X is irrational, then the cardinality of X is

(1) infinite (2) countably infinite

(3) 2 (4) 1

27. Suppose the matrix

$$A = \begin{pmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{pmatrix}$$

has a certain complex number $\lambda \ne 0$ as an eigenvalue. Which of the following numbers must also be an eigenvalue of A?

(1) $\lambda + 20$ (2) $\lambda - 20$

(3) $20 - \lambda$ (4) $-20 - \lambda$

28. Define $f : [0,1] \to [0,1]$ by $f(x) = \frac{2^k - 1}{2^k}$ for $x \in \left[\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^k - 1}{2^k}\right]$, $k \geq 1$.

Then f is a Riemann-integrable function such that

1. $\int_0^1 f(x)dx = \frac{2}{3}$
2. $\frac{1}{2} < \int_0^1 f(x)dx < \frac{2}{3}$
3. $\int_0^1 f(x)dx = 1$
4. $\frac{2}{3} < \int_0^1 f(x)dx < 1$

29. Let $A = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Then a Jordan canonical form of A is

(1) $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ (2) $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ (4) $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

30. Let $I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)}$, where $C_r = \{z \in \mathbb{C} : |z| = r\}$, $r > 0$. Then

1. $I_r = 2\pi i$ if $r \in (2,3)$
2. $I_r = \frac{1}{2}$ if $r \in (0,1)$
3. $I_r = -2\pi i$ if $r \in (1,2)$
4. $I_r = 0$ if $r > 3$

31. Let A be a 5×4 matrix with real entries such that the space of all solutions of the linear system $AX^t = [1, 2, 3, 4, 5]^t$ is given by $\{[1+2s, 2+3s, 3+4s, 4+5s]^t : s \in \mathbb{R}\}$. (Here M^t denotes the transpose of a matrix M.) Then the rank of A is equal to

(1) 4 (2) 3
(3) 2 (4) 1

32. Let A be a 3×3 matrix with real entries such that $\det(A) = 6$ and the trace of A is 0. If $\det(A + I) = 0$, where I denotes the 3×3 identity matrix, then the eigenvalues of A are

(1) $-1, 2, 3$ (2) $-1, 2, -3$
(3) $1, 2, -3$ (4) $-1, -2, 3$

33. Let $p(z,w) = \alpha_0(z) + \alpha_1(z)w + \ldots + \alpha_k(z)w^k$, where $k \geq 1$ and $\alpha_0, \ldots, \alpha_k$ are non-constant polynomials in the complex variable z. Then $\{(z,w) \in \mathbb{C} \times \mathbb{C} : p(z,w) = 0\}$ is

1. bounded with empty interior
2. unbounded with empty interior
3. bounded with nonempty interior
4. unbounded with nonempty interior

34. Let (X,d) be a metric space and let $A \subseteq X$. For $x \in X$, define $d(x,A) = \inf\{d(x,a) : a \in A\}$. If $d(x,A) = 0$ for all $x \in X$, then which of the following assertions must be true?

(1) A is compact (2) A is closed
(3) A is dense in X (4) $A = X$

35. Let n be a positive integer and let H_n be the space of all $n \times n$ matrices $A = (a_{ij})$ with entries in $\mathbb{R}$ satisfying $a_{ij} = a_{rs}$ whenever $i + j = r + s$ $(i, j, r, s = 1, \ldots, n)$. Then the dimension of H_n, as a vector space over $\mathbb{R}$, is

(1) n^2 (2) $n^2 - n + 1$

(3) $2n + 1$ (4) $2n - 1$

36. Let $\underset{\sim}{X}$ be a p dimensional random vector which follows $N(\underset{\sim}{0}, I_p)$ distribution and let A be a real symmetric matrix. Which of the following is true?

1. $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution if $A^2 = A$ but the converse is not true
2. If $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution then the degrees of freedom is equal to p
3. If $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution then the characteristic roots of A are either 0 or 1
4. If $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution then A is necessarily positive definite

37. For which of the following primes p, does the polynomial $x^4 + x + 6$ have a root of multiplicity > 1 over a field of characteristic p?

(1) $p = 2$ (2) $p = 3$

(3) $p = 5$ (4) $p = 7$

38. Suppose $X \sim N(0,1)$ and $Y \sim \chi_n^2$. Which of the following is always correct?

1. $X^2 + Y \sim \chi_{n+1}^2$
2. $\dfrac{X}{\sqrt{Y/n}} \sim t_n$
3. $E(X^2 + Y) = 1 + n$
4. $Var(X + Y) = 1 + 2n$

39. The number of multiples of 10^{44} that divide 10^{55} is

(1) 11 (2) 12

(3) 121 (4) 144

40. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $f_n : D \to \mathbb{C}$ be defined by $f_n(z) = \dfrac{z^n}{n}$ for $n = 1, 2, \ldots$. Then

1. the sequences $\{f_n(z)\}$ and $\{f_n'(z)\}$ converge uniformly on D
2. the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on D
3. the series $\sum_{n=1}^{\infty} f_n'(z)$ converges for each $z \in D$
4. the sequence $\{f_n''(z)\}$ does not converge unless $z = 0$

41. Let $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \quad a \le x \le b,$$

where $p(x)$ and $q(x)$ are real valued continuous functions on $[a,b]$. If x_0 and x_1, with $x_0 < x_1$, are consecutive zeros of $y_1(x)$ in (a, b), then

1. $y_1(x) = (x - x_0)q_0(x)$, where $q_0(x)$ is continuous on [a, b] with $q_0(x_0) \neq 0$
2. $y_1(x) = (x - x_0)^2 p_0(x)$, where $p_0(x)$ is continuous on [a, b] with $p_0(x_0) \neq 0$
3. $y_2(x)$ has no zeros in (x_0, x_1)
4. $y_2(x_0) = 0$ but $y_2'(x_0) \neq 0$

42. The number of group homomorphisms from the symmetric group S_3 to $\mathbb{Z}/6\mathbb{Z}$ is

(1) 1 (2) 2
(3) 3 (4) 6

43. The second order partial differential equation

$$\frac{(x-y)^2}{4}\frac{\partial^2 u}{\partial x^2}+(x-y)\sin\left(x^2+y^2\right)\frac{\partial^2 u}{\partial x\partial y}+\cos^2\left(x^2+y^2\right)\frac{\partial^2 u}{\partial y^2}+(x-y)\frac{\partial u}{\partial x}+\sin^2\left(x^2+y^2\right)\frac{\partial u}{\partial y}+u=0$$ is

1. Elliptic in the region $\left\{(x,y): x\neq y,\ x^2+y^2<\pi/6\right\}$
2. Hyperbolic in the region $\left\{(x,y): x\neq y,\ \frac{\pi}{4}<x^2+y^2<\frac{3\pi}{4}\right\}$
3. Elliptic in the region $\left\{(x,y): x\neq y,\ \frac{\pi}{4}<x^2+y^2<\frac{3\pi}{4}\right\}$
4. Hyperbolic in the region $\left\{(x,y): x\neq y,\ x^2+y^2<\frac{\pi}{4}\right\}$

44. Let $X_1, X_2, \ldots$ be i.i.d. nonnegative integer valued random variables with finite mean and let N be an independent positive integer valued random variable with finite mean. Let $G(s)=E\left(s^{X_1}\right)$ and $H(s)=E\left(s^N\right)$ for $|s|<1$ and $G'(s), H'(s)$ denote the derivatives of $G(s)$, $H(s)$ respectively.

For $Y=X_1+X_2+\ldots+X_N$, the mean of Y is given by

1. $H'(G(0))G'(0)$
2. $G'(H(0))H'(0)$
3. $\lim_{s\uparrow 1} H'(G(s))G'(s)$
4. $\lim_{s\uparrow 1} G'(H(1-s))H'(s)$

45. Let $u=u(x,y)$ be the complete integral of the PDE $\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}=xy$ passing through the points $(0,0,1)$ and $\left(0,1,\frac{1}{2}\right)$ in the $x-y-u$ space.

Then the value of $u(x,y)$ evaluated at $(-1,1)$ is

(1) 0 (2) 1
(3) 2 (4) 3

46. An aperiodic Markov chain with stationary transition probabilities on the state space $\{1,2,3,4,5\}$ must have

1. at least one null recurrent state
2. at least one positive recurrent state
3. at least one positive recurrent and at least one null recurrent state
4. at least one transient state

47. Consider a randomized (complete) block experiment involving v treatments and r replicates. Which of the following is true?

1. The variance of the best linear unbiased estimator (BLUE) of any elementary treatment contrast is σ^2/r, where σ^2 is the variance of an observation
2. The BLUE of a treatment contrast is uncorrelated with the BLUE of a block contrast
3. There are exactly $(r-1)$ linearly independent linear functions of the observations, each of which has zero expectation
4. If μ is the general mean and τ_i is the effect of the ith treatment $(i=1,2,\cdots,v)$, then $\mu+\tau_i$ is estimable for each $i, i=1,2,\cdots,v$

48. The complete integral of the PDE

$$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=xe^{x+y}$$

involving arbitrary functions ϕ_1 and ϕ_2 is

1. $\phi_1(y+x)+\phi_2(y+x)+\frac{1}{4}e^{x+y}$
2. $\phi_1(y+x)+x\phi_2(y+x)+\frac{(x-1)}{4}e^{x+y}$
3. $\phi_1(y-x)+\phi_2(y-x)+\frac{1}{4}e^{x+y}$
4. $\phi_1(y-x)+x\phi_2(y-x)+\frac{(x-1)}{4}e^{x+y}$

49. Suppose any two points $P(x_0,y_0)$, $Q(x_1,y_1)$, and a function $F(x,y,y')$ of three independent variables are given, where $y' \equiv \frac{dy}{dx}$. In order to find among all curves $y=y(x)$ joining P and Q that one which furnishes for the definite integral $I(y)=\int_{x_0}^{x_1} F(x,y(x),y'(x))\,dx$ the smallest value, which of the following assumption suffices:

1. Function F is of class C^1
2. Functional F is of class C^2 for all systems of values $x, y(x)$ and $y'(x)$ furnished by all of the admissible functions
3. Functional F is of class C^3 for all systems of values $x, y(x)$ and $y'(x)$ furnished by all of the admissible functions
4. It is enough to treat $y(x)$ and F to be of class C^1 only with respect to their arguments.

50. Let X_1, X_2 be a random sample from an exponential distribution with mean θ. Consider the problem of testing

$H_0:\theta=1$ against $H_1:\theta=\frac{1}{2}$. Suppose the p-values of the left-tailed tests based on (X_1+X_2) and X_1 are given by p_1 and p_2 respectively. Which of the following is always true?

1. $p_1 < p_2$
2. $p_1 > p_2$
3. $p_1 = p_2$
4. Nothing can be said about the relationship between p_1 and p_2

51. Consider the following linear programming problem

$Max\ Z = 2x_1 + x_2 + x_3$

subject to

$$x_1 - x_2 \le 10$$
$$2x_1 - x_2 \le 40$$
$$x_1 - x_3 \le 10$$
$$x_1,\ x_2,\ x_3 \ge 0.$$

The problem has

1. an unbounded solution
2. one optimal solution
3. more than one optional solutions
4. no feasible solution

52. Consider the equation

$\frac{dy}{dt} = (1+f^2(t))y(t),\ y(0)=1 : t \ge 0$

where f is a bounded continuous function on $[0,\infty)$. Then

1. This equation admits a unique solution $y(t)$ and further $\lim_{t\to\infty} y(t)$ exists and is finite
2. This equation admits two linearly independent solutions
3. This equation admits a bounded solution for which $\lim_{t\to\infty} y(t)$ does not exist
4. This equation admits a unique solution $y(t)$ and further, $\lim_{t\to\infty} y(t) = \infty$

53. Let a sample of size n be drawn from a finite population of size N using probability proportional to size sampling *with replacement*, the selection probabilities being

p_i, $1 \le i \le N$, $0 < p_i < 1$ and $\sum_{i=1}^{N} p_i = 1$.

Also, let y_i be the value of a study variable for the ith unit in the sample $(1 \le i \le n)$ and

$T = \sum_{i=1}^{n} y_i / (np_i)$. Which of the following is true?

1. The probability that the ith unit is included in the sample is p_i / n, $1 \le i \le N$
2. The variance of T is zero if $p_i = 1/N$ for each i, $1 \le i \le N$
3. The variance of T is zero if Y_i is proportional to p_i, where for $1 \le i \le N$, Y_i is the value of the study variable for the ith unit in the population
4. There is no unbiased estimator of the variance of T

54. For the computation of $\sqrt{x+1} - 1$ at $x = 1.2345678 \times 10^{-5}$ using a machine which keeps 8 significant digits, which of the following equivalent expressions would be best to use

1. $\sqrt{x+1} - 1$
2. $\left(1 - \frac{1}{\sqrt{x-1}}\right)\sqrt{x+1}$
3. $\frac{x}{\sqrt{x+1}+1}$
4. $\frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \ldots\ldots$

55. Let U denote the set of all $n \times n$ matrices A with complex entries such that A is unitary (i.e., $\overline{A}^t A = I_n$). Then U, as a topological subspace of $\mathbb{C}^{n^2}$, is

1. compact, but not connected
2. connected, but not compact
3. connected and compact
4. neither connected nor compact

56. Which of the following is correct?

1. The number of degrees of freedom of a two-particle (connected by a light rod of length L) system in a vertical plane, where one of the particles is constrained to move horizontally, is <u>two</u>
2. The number of degrees of freedom of a door swinging on its hinges is <u>three</u>
3. The number of independent generalized coordinates needed to specify the simple pendulum moving in a vertical plane is <u>two</u>
4. The number of scalar equations needed to determine the motion of an unconstrained N-particle system is <u>N</u>

57. A system has three components and the system works if at least two of the three components work. The life times of the components are independent and identically distributed exponential random variables with mean 1. If X denotes the life time of the system, then E(X) is

(1) 1 (2) 2/3
(3) 5/6 (4) 1/2

58. Let $X_1, \ldots, X_n$ be a random sample from the probability density function

$$f(x) = \lambda e^{-\lambda x}; x > 0,\ \lambda > 0,$$

where λ has the prior density

$$g(\lambda) = 9\lambda e^{-3\lambda}; \lambda > 0.$$

The Bayes' estimate of $P[1 < X_1 < 2]$ based on squared error loss function is

1. $\left[\frac{3+\Sigma x_i}{4+\Sigma x_i}\right]^n - \left[\frac{3+\Sigma x_i}{5+\Sigma x_i}\right]^n$
2. $\left[\frac{3+\Sigma x_i}{4+\Sigma x_i}\right]^{n+1} - \left[\frac{3+\Sigma x_i}{5+\Sigma x_i}\right]^{n+1}$

3. $\left[\dfrac{3+\Sigma x_i}{4+\Sigma x_i}\right]^{n+2} - \left[\dfrac{3+\Sigma x_i}{5+\Sigma x_i}\right]^{n+2}$

4. $\left[\dfrac{3+\Sigma x_i}{4+\Sigma x_i}\right]^{n+3} - \left[\dfrac{3+\Sigma x_i}{5+\Sigma x_i}\right]^{n+3}$

59. In a testing of hypothesis problem, the density of a sufficient statistic T is

$f(t,\theta)=\dfrac{\theta}{t^{\theta+1}}, t>1, \theta>0.$ The hypothesis $H_0:\theta=1$ against $H_1:\theta=2$ is to be tested and $T=2.5$ is observed. Then the p-value of the most powerful test is

(1) 0.05 (2) 0.5
(3) 0.6 (4) 0.4

60. Let Y_1, Y_2, Y_3 and Y_4 be four random variables such that

$E(Y_1)=\theta_1-\theta_3; E(Y_2)=\theta_1+\theta_2-\theta_3;$
$E(Y_3)=\theta_1-\theta_3; E(Y_4)=\theta_1-\theta_2-\theta_3,$

where $\theta_1,\theta_2,\theta_3$ are unknown parameters. Also assume that $Var(Y_i)=\sigma^2$, $i=1,2,3,4$. Then

1. $\theta_1,\theta_2,\theta_3$ are estimable
2. $\theta_1+\theta_3$ is estimable
3. $\theta_1-\theta_3$ is estimable and $\frac{1}{2}(Y_1+Y_3)$ is the best linear unbiased estimate of $\theta_1-\theta_3$
4. θ_2 is estimable

PART C

Unit - I

61. Which of the following real-valued functions f defined on $\mathbb{R}$ have the property that f is continuous and $f\circ f=f$?

(1) $f(x)=\begin{cases}|x| & \text{if } x\in[-1,1],\\ x^2 & \text{if } x\notin[-1,1]\end{cases}$

(2) $f(x)=\begin{cases}x & \text{if } x\in[0,1],\\ 1 & \text{if } x\geq 1,\\ 0 & \text{if } x\leq 0\end{cases}$

(3) $f(x)=\begin{cases}x & \text{if } x\in[-1,1],\\ 1 & \text{if } x\geq 1,\\ -1 & \text{if } x\leq -1\end{cases}$

(4) $f(x)=\begin{cases}1 & \text{if } x\in[-23,27],\\ 22+x & \text{if } x\leq -23,\\ -26+x & \text{if } x\geq 27\end{cases}$

62. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\int_0^\infty f(x)\,dx$ exists. Which of the following statements are correct?

1. if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} f(x)=0$
2. the limit $\lim_{x\to\infty} f(x)$ must exist and is zero
3. in case f is a nonnegative function, $\lim_{x\to\infty} f(x)$ must exist and is zero
4. in case f is a differentiable function, $\lim_{x\to\infty} f'(x)$ must exist and is zero

63. Let $f(r,\theta)=(r\cos\theta, r\sin\theta)$ for $(r,\theta)\in \mathbb{R}^2$ with $r\neq 0$. Which of the following statements are correct? (Here Df denotes the derivative of f).

1. the linear transformation $Df(r,\theta)$ is not zero for any $(r,\theta)\in \mathbb{R}^2$ with $r\neq 0$
2. f is one-one on $\{(r,\theta)\in \mathbb{R}^2 : r\neq 0\}$
3. for any $(r,\theta)\in \mathbb{R}^2$ with $r\neq 0$, f is one-one on a neighborhood of (r,θ)
4. $Df(r,\theta)=r^2 I$ for any $(r,\theta)\in \mathbb{R}^2$ with $r\neq 0$

64. The map $L:\mathbb{R}^2\to\mathbb{R}^2$ given by $L(x,y)=(x,-y)$ is

(1) differentiable everywhere on $\mathbb{R}^2$ (2) differentiable only at (0,0)

(3) $DL(0,0)=L$ (4) $DL(x,y)=L$ for all $(x,y)\in\mathbb{R}^2$

65. It is given that the series $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent and $\sum_{n=1}^{\infty} a_n=0$. Denote by s_k the partial sum $\sum_{n=1}^{k} a_n$, $k=1,2,\cdots$ Then

1. $s_k=0$ for infinitely many k
2. $s_k>0$ for infinitely many k, and $s_k<0$ for infinitely many k
3. it is possible that $s_k>0$ for all k
4. it is possible that $s_k>0$ for all but a finite number of values of k

66. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases} x^2 & \text{if } x<0, \\ 2x+x^2 & \text{if } x\geq 0 \end{cases}$

Then which of the following statements are correct?

(1) $f''(x)=2$ for all $x\in\mathbb{R}$ (2) $f''(0)$ does not exist

(3) $f'(x)$ exists for each $x\neq 0$ (4) $f'(0)$ does not exist

67. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that f' is bounded. Given a closed and bounded interval $[a,b]$, and a partition $P=\{a_0=a<a_1<\cdots<a_n=b\}$ of $[a,b]$, let $M(f,P)$ and $m(f,P)$ denote, respectively, the upper Riemann sum and the lower Riemann sum of f with respect to P. Then

1. $\left|M(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)\sup\{|f(x)| : x\in[a,b]\}$

2. $\left|m(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)\inf\{|f(x)| : x\in[a,b]\}$

3. $\left|M(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)^2\sup\{|f'(x)| : x\in[a,b]\}$

4. $\left|m(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)^2\inf\{|f'(x)| : x\in[a,b]\}$

68. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be the function defined by $f(x)=x\|x\|^2$ for $x\in\mathbb{R}^n$. Which of the following statements are correct?

(1) $(Df)(0)=0$ **(2)** $(Df)(x)=0$ for all $x\in\mathbb{R}^n$

(3) f is one-one **(4)** f has an inverse

69. Consider the quadratic form $q(x,y,z)=4x^2+y^2-z^2+4xy-2xz-yz$ over $\mathbb{R}$. Which of the following statements about the range of values taken by q as x,y,z vary over $\mathbb{R}$, are true?

1. range contains $[1,\infty)$
2. range is contained in $[0,\infty)$
3. range = $\mathbb{R}$
4. range is contained in $[-N,\infty)$ for some large natural number N depending on q

70. Consider a matrix $A=(a_{ij})_{n\times n}$ with integer entries such that

$$a_{ij}=0 \text{ for } i>j \text{ and } a_{ii}=1 \text{ for } i=1,\ldots,n.$$

Which of the following properties must be true?

1. A^{-1} exists and it has integer entries
2. A^{-1} exists and it has some entries that are not integers
3. A^{-1} is a polynomial function of A with integer coefficients
4. A^{-1} is not a power of A unless A is the identity matrix

71. Let J be the 3×3 matrix all of whose entries are 1. Then:

1. 0 and 3 are the only eigenvalues of A
2. J is positive semidefinite, i.e., $\langle Jx,x\rangle\geq0$ for all $x\in\mathbb{R}^3$
3. J is diagonalizable
4. J is positive definite, i.e., $\langle Jx,x\rangle>0$ for all $x\in\mathbb{R}^3$ with $x\neq0$

72. Let A,B be complex $n\times n$ matrices. Which of the following statements are true?

1. If A, B, and $A+B$ are invertible, then $A^{-1}+B^{-1}$ is invertible
2. If A, B, and $A+B$ are invertible, then $A^{-1}-B^{-1}$ is invertible
3. If AB is nilpotent, then BA is nilpotent
4. Characteristic polynomials of AB and BA are equal if A is invertible

73. Let ω be a complex number such that $\omega^3=1$, but $\omega\neq1$. If

$$A=\begin{bmatrix}1 & \omega & \omega^2\\ \omega & \omega^2 & 1\\ \omega^2 & \omega & 1\end{bmatrix},$$

then which of the following statements are true?

1. A is invertible
2. $\text{rank}(A)=2$
3. 0 is an eigenvalue of A
4. there exist linearly independent vectors $v, w\in\mathbb{C}^3$ such that $Av=Aw=0$

74. Let A be a 4×4 matrix with real entries such that $-1, 1, 2, -2$ are its eigenvalues. If $B=A^4-5A^2+5I$, where I denotes the 4×4 identity matrix, then which of the following statements are correct?

(1) $\det(A+B)=0$ (2) $\det(B)=1$

(3) trace of $A-B$ is 0 (4) trace of $A+B$ is 4

75. Let $M_2(\mathbb{R})$ denote the set of 2×2 real matrices. Let $A\in M_2(\mathbb{R})$ be of trace 2 and determinant -3. Identifying $M_2(\mathbb{R})$ with $\mathbb{R}^4$, consider the linear transformation $T:M_2(\mathbb{R})\to M_2(\mathbb{R})$ defined by $T(B)=AB$. Then which of the following statements are true?

(1) T is diagonalizable (2) 2 is an eigenvalue of T

(3) T is invertible (4) $T(B)=B$ for some $0\neq B$ in $M_2(\mathbb{R})$

76. Let A be a 2×2 non-zero matrix with entries in $\mathbb{C}$ such that $A^2=0$. Which of the following statements must be true?

1. PAP^{-1} is diagonal for some invertible 2×2 matrix P with entries in $\mathbb{R}$
2. A has two distinct eigenvalues in $\mathbb{C}$
3. A has only one eigenvalue in $\mathbb{C}$ with multiplicity 2
4. $Av=v$ for some $v\in\mathbb{C}^2, v\neq 0$

77. Consider the linear transformation $T:\mathbb{R}^7\to\mathbb{R}^7$ defined by

$$T(x_1,x_2,\ldots,x_6,x_7)=(x_7,x_6,\ldots,x_2,x_1).$$

Which of the following statements are true?

1. the determinant of T is 1
2. there is a basis of $\mathbb{R}^7$ with respect to which T is a diagonal matrix
3. $T^7=I$
4. The smallest n such that $T^n=I$ is even

78. Let λ,μ be distinct eigenvalues of a 2×2 matrix A. Then, which of the following statements must be true?

1. A^2 has distinct eigenvalues
2. $A^3=\dfrac{\lambda^3-\mu^3}{\lambda-\mu}A-\lambda\mu(\lambda+\mu)I$
3. trace of A^n is $\lambda^n+\mu^n$ for every positive integer n
4. A^n is not a scalar multiple of identity for any positive integer n

Unit - II

79. Let f be an entire function such that $\lim\limits_{|z|\to\infty}|f(z)|=\infty$. Then

1. $f\left(\frac{1}{z}\right)$ has an essential singularity at 0.
2. f cannot be a polynomial
3. f has finitely many zeros
4. $f\left(\frac{1}{z}\right)$ has a pole at 0

80. Let f be a holomorphic function on $D=\{z\in\mathbb{C} : |z|<1\}$ such that $|f(z)|\le 1$ Define $g:D\to\mathbb{C}$ by

$$g(z)=\begin{cases}\dfrac{f(z)}{z} & \text{if } z\in D,\ z\neq 0,\\ f'(0) & \text{if } z=0.\end{cases}$$

Which of the following statements are true?

(1) g is holomorphic on D

(2) $|g(z)|\le 1$ for all $z\in D$

(3) $|f'(z)|\le 1$ for all $z\in D$

(4) $|f'(0)|\le 1$

81. Let f,g be holomorphic functions defined on $A\cup D$, where

$$A = \left\{z\in\mathbb{C} : \frac{1}{2}<|z|<1\right\} \text{ and}$$
$$D = \{z\in\mathbb{C} : |z-2|<1\}.$$

Which of the following statements are correct?

1. if $f(z)g(z)=0$ for all $z\in A\cup D$, then either $f(z)=0$ for all $z\in A$ or $g(z)=0$ for all $z\in A$
2. if $f(z)g(z)=0$ for all $z\in D$, then either $f(z)=0$ for all $z\in D$ or $g(z)=0$ for all $z\in D$
3. if $f(z)g(z)=0$ for all $z\in A$, then either $f(z)=0$ for all $z\in A$ or $g(z)=0$ for all $z\in A$
4. if $f(z)g(z)=0$ for all $z\in A\cup D$, then either $f(z)=0$ for all $z\in A\cup D$, or $g(z)=0$ for all $z\in A\cup D$,

82. Let $\mathbb{Z}[i]$ denote the ring of Gaussian integers. For which of the following values of n is the quotient ring $\mathbb{Z}[i]/n\mathbb{Z}[i]$ an integral domain?

(1) 2 (2) 13 (3) 19 (4) 7

83. Let U be an open subset of $\mathbb{C}$ containing $\{z\in\mathbb{C}:|z|\le 1\}$ and let $f:U\to\mathbb{C}$ be the map defined by

$$f(z)=e^{i\psi}\frac{z-a}{1-\bar{a}z} \text{ for } a\in D, \text{and } \psi\in[0,2\pi].$$

Which of the following statements are true?

(1) $|f(e^{i\theta})|=1$ for $0\le\theta\le 2\pi$

(2) f maps $\{z\in\mathbb{C} : |z|<1\}$ onto itself

(3) f maps $\{z\in\mathbb{C} : |z|\le 1\}$ into itself

(4) f is one-one

84. Which of the following integral domains are Euclidean domains?

1. $\mathbb{Z}\left[\sqrt{-3}\right]=\left\{a+b\sqrt{-3} : a,b\in\mathbb{Z}\right\}$
2. $\mathbb{Z}[x]$
3. $\mathbb{R}\left[x^2,x^3\right]=\left\{f=\sum_{i=0}^{n}a_i x^i\in\mathbb{R}[x] : a_1=0\right\}$
4. $\left(\frac{\mathbb{Z}[x]}{(2,x)}\right)[y]$ where x, y are independent variables and $(2, x)$ is the ideal generated by 2 and x

85. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function and let $g:\mathbb{C}\to\mathbb{C}$ be defined by $g(z)=f(z)-f(z+1)$ for $z\in\mathbb{C}$. Which of the following statements are true?

1. if $f\left(\frac{1}{n}\right)=0$ for all positive integers n, then f is a constant function
2. if $f(n)=0$ for all positive integers n, then f is a constant function
3. if $f\left(\frac{1}{n}\right)=f\left(\frac{1}{n}+1\right)$ for all positive integers n, then g is a constant function
4. if $f(n)=f(n+1)$ for all positive integers n, then g is a constant function

86. For $x\in\mathbb{R}^n$, let $B(x,r)$ denote the closed ball in $\mathbb{R}^n$ (with the Euclidean norm) of radius r centered at x. Write $B=B(0,1)$. If $f, g:\mathrm{B}\to\mathbb{R}^n$ are continuous functions such that $f(x)\neq g(x)$ for all $x\in B$, then

1. $f(B)\cap g(B)=\varnothing$
2. there exists $\varepsilon>0$ such that $\|f(x)-g(x)\|>\varepsilon$ for all $x\in B$
3. there exists $\varepsilon>0$ such that $B(f(x),\varepsilon)\cap B(g(x),\varepsilon)=\varnothing$ for all $x\in B$
4. $\mathrm{Int}(f(B))\cap\mathrm{Int}(g(B))=\varnothing$, where Int(E) denotes the interior of a set E.

87. Let G be the Galois group of the splitting field of x^5-2 over $\mathbb{Q}$. Then, which of the following statements are true?

(1) G is cyclic
(2) G is non-abelian
(3) the order of G is 20
(4) G has an element of order 4

88. For which of the following values of n, does the finite field F_{5^n} with 5^n elements contain a non-trivial 93^{rd} root of unity?

(1) 92 (2) 30 (3) 15 (4) 6

89. Let X denote the product of countably many copies of $[0,1]$. We let X_1 denote the set X equipped with the box topology and let X_2 denote the set X equipped with the product topology. Then

(1) X_1 is compact and separable (2) X_2 is compact and separable

(3) X_1 and X_2 are both compact (4) Neither X_1 nor X_2 is separable

90. Which of the following numbers can be orders of permutations σ of 11 symbols such that σ does not fix any symbol?

(1) 18 (2) 30 (3) 15 (4) 28

Unit - III

91. Consider the functional

$$I(y)=\int_{1/2}^{x_1}\sqrt{1+y'^2}\,dx$$

subject to the condition that the left end of the extremal is fixed at the point $\left(\frac{1}{2},\frac{1}{4}\right)$ and the right end (x_1, y_1) be movable along the straight line $y=x-5$. Let C_1 and C_2 be arbitrary constants. Then

1. the general solution of the Euler's equation satisfies $2C_1+4C_2=1$
2. the transversality condition that holds at the right end yields
$$\sqrt{\left(1+C_1^2\right)}+\left(1-C_1\right)\frac{C_1}{\sqrt{1+C_1^2}}=0$$
3. the relation $C_1x_1+C_2=x_1-5$ is true
4. the extremal is $y=-x+\frac{3}{4}$

92. For the integral equation $u(x)=f(x)+\lambda\int_a^b K(x,t)u(t)\,dt$ to have a continuous solution in the interval $a\le x\le b$, which of the following assumptions are necessary?

1. $K(x,t)\neq 0$, is real and continuous in the region $a\le x\le b$, $a\le t\le b$ with $|K(x,t)|\le M$
2. $f(x)\neq 0$, is real and continuous in the interval $a\le x\le b$,

3. λ is a constant
4. $|\lambda| < \frac{1}{M(b-a)}$

93. Suppose that the potential energy $V(q_1, q_2)$ of a system with generalized coordinates q_1 and q_2 has a minimum at $q_1 = q_2 = 0$ and that $V(0,0) = 0$. Then

1. the approximate V near (0, 0) is a homogeneous quadratic form in q_1 and q_2 assuming that the terms in powers three (or higher) in the small quantities q_1 and q_2 are negligible
2. if (0,0) is also a minimum point of approximate V, then the quadratic form in the option 1 given above is positive definite
3. the requirement in the option 2 given above implies that the approximate V takes positive values except when $q_1 = q_2 = 0$
4. the requirement in the option 2 given above that (0,0) is a minimum point of approximate V implies that $q_1 = q_2 = 0$ yields a strict minimum of the exact V

94. The integral equation

$y(x) = 1 + \lambda \int_0^{\pi/2} \cos(x-t)\, y(t)\, dt$ has

1. a unique solution for $\lambda \neq 4/(\pi+2)$
2. a unique solution for $\lambda \neq 4/(\pi-2)$
3. no solution for $\lambda = 4/(\pi+2)$, but the corresponding homogeneous equation has non-trivial solutions
4. no solution for $\lambda = 4/(\pi-2)$, but the corresponding homogenous equation has non-trivial solutions

95. Consider the Euler's equations of motion for the free motion of a body relative to its centre of mass O and principal axes OX_1, OX_2, OX_3. If the body is in motion with OX_3 as the axis of symmetry with $\bar{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\bar{L} = (L_1, L_2, L_3)$ as the angular velocity vector and the angular momentum vector, respectively, then

1. ω_3 = constant
2. ω_1 and ω_2 satisfy the simple harmonic motion equation
3. $\bar{e}_3$, the unit vector along OX_3, $\bar{\omega}$, and $\bar{L}$ lie in the same plane
4. L_3 = constant

96. Consider the first order system of linear equations

$\frac{dX}{dt} = AX;\ A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix};\ X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$

Then

1. the coefficient matrix A has a repeated eigenvalue $\lambda = 1$
2. there is only one linearly independent eigenvector $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
3. the general solution of the ODE is $(aX_1 + bX_2)e^t$, where a, b are arbitrary constants and $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} t \\ \frac{1}{2} - t \end{bmatrix}$
4. the vectors X_1 and X_2 in the option 3 given above are linearly independent

97. Consider the interpolation data given below:

x	1	½	3
y	3	−10	2

The interpolating polynomial corresponding to this data is given by

1. $p(x) = -3\left(x - \frac{1}{2}\right)(x-3) - 8(x-1)(x-3) + \frac{2}{5}(x-1)\left(x - \frac{1}{2}\right)$
2. $q(x) = 3 + 26(x-1) + \frac{-53}{5}(x-1)\left(x - \frac{1}{2}\right)$
3. $r(x) = \frac{-53}{5}x^2 + \frac{419}{10}x + \frac{-283}{10}$
4. $p(x)q(x) + r(x)$

98. The Green's function G(x,t) of the boundary value problem

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 1, \quad y(0) = y(1) = 0$$

is $G(x,t) = \begin{cases} f_1(x,t), & \text{if } x \le t \\ f_2(x,t), & \text{if } t \le x, \end{cases}$

where

1. $f_1(x,t) = -\frac{1}{2}t(1-x^2), \; f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$
2. $f_1(x,t) = -\frac{1}{2x}t^2(1-x^2), \; f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$
3. $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2), \; f_2(x,t) = -\frac{1}{2}t(1-x^2)$
4. $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2), \; f_2(x,t) = -\frac{1}{2x}t^2(1-x^2)$

99. Let $A=\begin{pmatrix}1.01 & 0.99\\ 0.99 & 1.01\end{pmatrix}$, $b=\begin{pmatrix}2\\2\end{pmatrix}$ and x_0 be the unique solution of the equation $Ax=b$,

Let $\hat{x}_1$ and $\hat{x}_2$ be the two approximate solutions $\begin{pmatrix}1.01\\1.01\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$.

Finally, let $r_1 = Ax_0 - A\hat{x}_1 = b - A\hat{x}_1$ and $r_2 = Ax_0 - A\hat{x}_2 = b - A\hat{x}_2$ be the corresponding residues. Which of the following is/are correct?

1. $\hat{x}_1$ is a good approximation to x_0 and r_1 is small
2. $\hat{x}_1$ is not a good approximation to x_0 but r_1 is small
3. $\hat{x}_2$ is a good approximation to x_0 and r_2 is small
4. $\hat{x}_2$ is not a good approximation to x_0 but r_2 is small

100. Let $u(x,t)$ be the solution of the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0<x<\infty,\ t>0$$

$$u(x,0)=\cos\left(\frac{\pi x}{2}\right),\ 0\le x<\infty$$

$$\frac{\partial u}{\partial t}(x,0)=0,\ 0\le x<\infty,$$

$$\frac{\partial u}{\partial x}(0,t)=0,.\ t\ge 0.$$

Then

(1) The value of $u(2,2)=-1$

(2) The value of $u(2,2)=1$

(3) The value of $u\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{\sqrt{2}}$

(4) The value of $u\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}$

101. The differential equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

satisfying the initial condition $y=xg(x)$, $u=f(x)$ with

1. $f(x)=2x$, $g(x)=1$, has no solution
2. $f(x)=2x^2$, $g(x)=1$, has infinite number of solutions
3. $f(x)=x^3$, $g(x)=x$, has a unique solution
4. $f(x)=x^4$, $g(x)=x$, has a unique solution

102. Let $\frac{d^2y}{dx^2} - q(x)y = 0, \quad 0 \le x < \infty,$

$y(0) = 1, \frac{dy}{dx}(0) = 1,$ where $q(x)$ is a positive monotonically increasing continuous function. Then

1. $y(x) \to \infty$ as $x \to \infty$
2. $\frac{dy}{dx} \to \infty$ as $x \to \infty$
3. $y(x)$ has finitely many zeros in $[0, \infty)$
4. $y(x)$ has infinitely many zeros in $[0, \infty)$

Unit - IV

103. Consider a multiple linear regression model with r regressors, $r \ge 1$ and the response variable Y. Suppress $\hat{Y}$ is the fitted value of Y, R^2 is the coefficient of determination and R^2_{adj} is the adjusted coefficient of determination. Then

1. R^2 always increases if an additional regressor is included in the model
2. R^2_{adj} always increases if an additional regressor is included in the model
3. $R^2 \ge R^2_{\text{adj}}$ for all r
4. correlation coefficient between Y and $\hat{Y}$ is always non-negative

104. Let $X_1, X_2 \cdots X_n$ be independent and identically distributed random variables with common mean μ and finite variance σ^2. Define $S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then which of the following is/are true?

(1) S_n is unbiased for estimating σ (2) S_n^2 is unbiased for estimating σ^2

(3) S_n is consistent for estimating σ (4) S_n^2 is consistent for estimating σ^2

105. The statement

"X_n converges to X in distribution"

is equivalent to

1. $\limsup_{n\to\infty} P[X_n \le x] \le P[X \le x]$ for all real x
2. $\liminf_{n\to\infty} P[X_n < x] \ge P[X < x]$ and $\liminf_{n\to\infty} P[X_n > x] \ge P[X > x]$ for all real x

3. $E\left[g(X_n)\right] \to \left[g(X)\right]$ for all bounded continuous functions g

4. $E\left[g(X_n)\right] \to E\left[g(X)\right]$ for all uniformly continuous functions g

106. An experiment consists of independent trials. At the n th trial a number is chosen uniformly at random from $\{1, 2, \ldots, n\}$. A random variable T_n is defined as follows

$$T_n = \begin{cases} 1, & \text{if } n \text{ ischosen at the } n\text{th trial} \\ 0, & \text{otherwise.} \end{cases}$$

Then

1. $P(T_n = 1 \text{ for infinitely many } n \geq 1) = 1$
2. $P(T_n = 0 \text{ for infinitely many } n \geq 1) = 0$
3. $P\left(T_n = 1 \text{ and } T_{n+1} = 0 \text{ for infinitely many } n \geq 1\right) = 1$
4. $P\left(T_n = 1 \text{ and } T_{n+1} = 1 \text{ for infinitely many } n \geq 1\right) = 0$

107. Let $\left(X_{n1}, X_{n2}, X_{n3}, X_{n4}\right)$ have a multinomial distribution with probability mass function

$$P[X_{n1} = k_1, X_{n2} = k_2, X_{n3} = k_3, X_{n4} = k_4] = \frac{n!}{k_1!k_2!k_3!k_4!}\left(\tfrac{1}{n}\right)^{k_1}\left(\tfrac{1}{n}\right)^{k_2}\left(\tfrac{n-2}{2n}\right)^{k_3}\left(\tfrac{n-2}{2n}\right)^{k_4},$$

where k_1, k_2, k_3, k_4 are non-negative integers and $k_1 + k_2 + k_3 + k_4 = n$. Then

1. X_{n1} converges in distribution to a Poisson random variable with mean 1
2. $\left(X_{n1}, X_{n2}\right)$ converges in distribution to $\left(X_1, X_2\right)$ where X_1 and X_2 are <u>not</u> independent
3. $\left(X_{n1}, X_{n3}\right)$ converges in distribution to $\left(X_1, X_3\right)$ where X_1 and X_3 are independent
4. $\left(X_{n1}, X_{n2}, X_{n3}\right)$ does not converge in distribution

108. Let $X_1, X_2, \ldots$ be i.i.d random variables with mean 0 and variance 2 and let $Y_i = \dfrac{X_i + X_{i+1}}{2}$ for $i \geq 1$. Then the limiting distribution of

1. $\dfrac{1}{\sqrt{2n}}\left(Y_1 + Y_2 + \cdots + Y_n\right)$ is normal with mean 0 and variance 1
2. $\dfrac{1}{\sqrt{n}}\left(Y_1 + Y_3 + Y_5 + \cdots + Y_{2n-1}\right)$ is normal with mean 0 and variance 1
3. $\dfrac{1}{\sqrt{n}}\left(Y_2 + Y_4 + Y_6 + \cdots + Y_{2n}\right)$ is normal with mean 0 and variance 1
4. $\dfrac{1}{\sqrt{n}}\left(Y_1 + Y_{2^1} + Y_{2^2} + \cdots + Y_{2^{n-1}}\right)$ is normal with mean 0 and variance 1

109. Consider the following primal problem

Max $Z = 5x_1 + 12x_2 + 4x_3$

subject to

$x_1 + 2x_2 + x_3 \le 10$

$2x_1 - x_2 + 3x_3 \le 8$

x_1 unrestricted, $x_2, x_3 \ge 0$.

Then

1. the objective function of the dual problem is Min $W = 10y_1 + 8y_2$
2. two of the constraints of the dual problem are $y_1 + 2y_2 = 5$, $2y_1 - y_2 \ge 12$
3. one of the constraint of the dual problem is $2y_1 + 3y_2 \ge 4$
4. for a pair of feasible primal and dual solutions we have $v_1 \ge v_2$ where v_1 is the value of the objective function in the maximization problem and v_2 is the value of the objective function in the minimization problem

110. For a random variable X with probability density function

$$f(x) = \left(1 + \alpha x^{\alpha-1}\right) e^{-\left(x + x^{\alpha}\right)}, x > 0, \alpha > 0,$$

the hazard function can be

(1) constant for some α

(2) an increasing function for some α

(3) a decreasing function for some α

(4) a bathtub-shaped function for some α

111. Let d be a balanced incomplete block design with usual parameters v, b, r, k, λ and let $\tau_i \, (1 \le i \le v)$ be the effect of the i th treatment. Which of the following is/are true?

1. d is connected whenever $k \ge 2$
2. Let $p_i \; (1 \le i \le v)$ be real numbers satisfying $\sum_{i=1}^{v} p_i = 0$ and $\sum_{i=1}^{v} p_i^2 = 1$. The variance of the best linear unbiased estimator of $\sum_{i=1}^{v} p_i \tau_i$, under d, does not depend on the p_i's
3. For $1 \le i \le v$, let p_i and q_i be real numbers satisfying $\sum_{i=1}^{v} p_i = 0 = \sum_{i=1}^{v} q_i$ and $\sum_{i=1}^{v} p_i q_i = 0$. The covariance between the best linear unbiased estimators of $\sum_{i=1}^{v} p_i \tau_i$ and $\sum_{i=1}^{v} q_i \tau_i$, under d, is zero
4. For d, the inequality $b \ge v + r - k$ holds

112. Consider a finite population of $N = nk$ units, where $n(\geq 2)$, $k(\geq 2)$ are integers. A linear systematic sample of n units is drawn from the population. Which of the following is/are true?

1. The probability that the ith unit is included in the sample is $1/k$, $i = 1, 2, \cdots, N$
2. If π_{ij} denotes the probability of inclusion of the units i and j in the sample, $i \neq j$, $i, j = 1, 2, \ldots, N$, then π_{ij} is zero for some pairs (i, j)
3. An unbiased estimator of the population mean of a study variable is the sample mean
4. There always exists an unbiased estimator of the variance of the sample mean

113. Let X_1, X_2, X_3, X_4 and Y_1, Y_2, Y_3 be two independent random samples from the continuous distributions $F(x)$ and $F(x-\Delta)$ respectively.
Define Rank $(X_i) = R_i$, $i = 1,2,3,4$ among $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3$. If the observed value of $\sum_{i=1}^{4} R_i$ is 11 and the Wilcoxon rank-sum test is used for testing $H_0: \Delta = 0$ against $H_1: \Delta > 0$, then which of the following is/are true?

(1) P-value < 0.06
(2) Reject H_0 at 5% level of significance
(3) Accept H_0 at 5% level of significance
(4) Accept H_0 at 1% level of significance

114. $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ are independent and identically distributed random vectors where X_1 is normally distributed with mean 0 and variance 1 and $P[Y_1 = 2] = P[Y_1 = -2] = \frac{1}{2}$. Further, X_i and Y_i are independently distributed for $i = 1, \ldots, n$ and $Z_n = X_1Y_1 - X_2Y_2 + \cdots + (-1)^{n-1} X_nY_n$. Then

(1) Z_n is normally distributed for each n
(2) Z_n is symmetrically distributed abou

(3) $V\left(\frac{Z_n}{\sqrt{n}}\right) = 4$
(4) $V\left(\frac{Z_n}{\sqrt{n}}\right) = 2$

115. Let Q_n denote the length of a queue at time n in an $M/M/1$ queue with arrival rate $\lambda > 0$, service rate $\mu > 0$ and $\rho = \frac{\lambda}{\mu}$. Which of the following is/are true?

1. If $\lambda < \mu$, then $\lim_{n\to\infty} P(Q_n = k) = (1-\rho)\rho^k$, $k \geq 0$
2. If $\lambda = \mu$, then $\lim_{n\to\infty} P(Q_n = k) = \frac{1}{2^{k+1}}$, $k \geq 0$
3. If $\lambda > \mu$, then $\lim_{n\to\infty} P(Q_n = k) = (1-\rho)\rho^k$, $k \geq 0$
4. If $\lambda = \mu$, then $\lim_{n\to\infty} P(Q_n = k) = 0$, $k \geq 0$

116. Let P be the stationary transition probability matrix of the Markov Chain $\{X_n, n \geq 0\}$, which is irreducible and every state has period 2. Further suppose that the Markov chain $\{Y_n, n \geq 0\}$ on the same state space has transition probability matrix P^2. Both the chains are assumed to have the same initial distribution. Then

1. $P[X_0 = Y_0, X_2 = Y_1] = 1$
2. all states of the chain $\{Y_n, n \geq 0\}$ are aperiodic
3. the chain $\{Y_n, n \geq 0\}$ is irreducible
4. if a state is recurrent for the chain $\{X_n, n \geq 0\}$, then it is also recurrent for the chain $\{Y_n, n \geq 0\}$

117. $X_1, X_2, \cdots, X_n$ are independent and identically distributed random variables with a common p.d.f. parametrized by $\theta > 0$ given by

$$f(x) = \begin{cases} \dfrac{\theta}{x^2}, & x \geq \theta \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_{min} = \min\{X_1, \dots, X_n\}$ and $X_{max} = \max\{X_1, \dots, X_n\}$. Then

1. X_{min} is sufficient for θ
2. $\dfrac{1}{\prod_{i=1}^{n} X_i^2}$ is sufficient for θ
3. X_{max} is maximum likelihood estimator of θ
4. X_{min} is maximum likelihood estimator of θ

118. Let $\underset{\sim}{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ be the stationary distribution for a Markov chain on the state space $\{1,2,3,4\}$ with transition probability matrix P. Suppose that the states 1 and 2 are transient and the states 3 and 4 form a communicating class. Which of the following is/are true?

(1) $\underset{\sim}{\mu} P^3 = \underset{\sim}{\mu} P^5$

(2) $\mu_1 = 0$ and $\mu_2 = 0$

(3) $\mu_3 + \mu_4 = 1$

(4) One of μ_3 and μ_4 is zero.

119. Suppose $\underset{\sim}{X}$ is a p-dimensional random vector with variance-covariance matrix Σ. If $\underset{\sim}{P}_1, \cdots \underset{\sim}{P}_p$ represent p orthonormal eigenvectors of Σ corresponding to the eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_p \geq 0$ respectively, then which of the following is/are true?

(1) First principal component is $\underset{\sim}{P}_1^T \underset{\sim}{X}$

(2) $Var\left(\underset{\sim}{P}_1^T \underset{\sim}{X}\right) = \lambda_1$

(3) $\underset{\sim}{P}_1^T \underset{\sim}{X}$ and $\underset{\sim}{P}_2^T \underset{\sim}{X}$ are correlated

(4) $Tr(\Sigma) = \sum_{i=1}^{p} \lambda_i$

120. Consider the problem of testing $H_0 : X \sim p_0$ against $H_1 : X \sim p_1$ where p_0 and p_1 are given by

x	0	1	2	3	4	5
$p_0(x) = P_{H_0}(X = x)$	0.05	0.02	0.03	0.05	0.35	0.5
$\frac{p_1(x)}{p_0(x)} = \frac{P_{H_1}(X = x)}{P_{H_0}(X = x)}$	2	1.5	1.5	3	0.4	1.07

Define three tests ϕ_1, ϕ_2 and ϕ_3 such that

$$\phi_1(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \begin{cases} 1, & \text{if } x = 3 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\phi_3(x) = \begin{cases} 1, & \text{if } x = 1 \text{ or } 2 \\ 0, & \text{otherwise.} \end{cases}$$

Then which of the following is/are true?

1. ϕ_1 is a most powerful test at level 0.05 for testing H_0 against H_1
2. ϕ_2 is a most powerful test at level 0.05 for testing H_0 against H_1
3. ϕ_3 is a most powerful test at level 0.05 for testing H_0 against H_1
4. ϕ_3 is unbiased at level 0.05 for testing H_0 against H_1