MATHEMATICS

Time: 2 hours *Marks: 60*

1. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$ then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$. **[2]**

2. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum. **[2]**

3. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^TA = I$, then find the value of $a^3 + b^3 + c^3$. **[2]**

4. Prove that $2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \ldots (-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$. **[2]**

5. If f is an even function then prove that $\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$. **[2]**

6. For a student to qualify, he must pass at least two out of three exams. The probability that he will pass the 1st exam is p. If he fails in one of the exams then the probability of his passing in the next exam is $\frac{p}{2}$ otherwise it remains the same. Find the probability that he will qualify. **[2]**

7. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point P(6, 8) to the circle and the chord of contact is maximum. **[2]**

8. Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$. **[2]**

9. A is targeting to B, B and C are targeting to A. Probability of hitting the target by A, B and C are $\frac{2}{3}$, $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If A is hit then find the probability that B hits the target and C does not. **[2]**

10. If a function $f : [-2a, 2a] \rightarrow R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at $x = a$ is 0 then find the left hand derivative at $x = -a$. **[2]**

11 Using the relation $2(1 - \cos x) < x^2, x \neq 0$ or otherwise, prove that $\sin(\tan x) \geq x, \forall x \in \left[0, \frac{\pi}{4}\right]$ **[4]**

12. If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either a = b = c or a, b, $\frac{c}{2}$

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SOLUTIONS

1. To prove $|1 - z_1\bar{z}_2| < |z_1 - z_2|$

$\Leftrightarrow (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

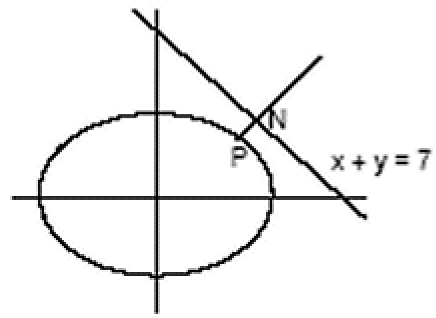
$\Leftrightarrow 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + |z_1|^2|z_2|^2 < |z_1|^2 - z_2\bar{z}_1 - z_1\bar{z}_2 + |z_2|^2$

$\Leftrightarrow (1 - |z_1|^2) - |z_2|^2 (1 - |z_1|^2) < 0$

$\Leftrightarrow (1 - |z_2|^2)(1 - |z_1|^2) < 0$

Which is obvious as $|z_1| < 1 < |z_2|$.

2. $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$



shortest distance exists along the common normal

Slope of normal at P

$= \frac{\sqrt{6}\sec\theta}{\sqrt{3}\,\mathrm{cosec}\,\theta} = \sqrt{2}\tan\theta = 1$

so $\cos\theta = \sqrt{\frac{2}{3}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$

Hence $P \equiv (2, 1)$.

3. $A^TA = I$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2 + b^2 + c^2 = 1$ ....(1)

and $ab + bc + ca = 0$ ....(2)

Now $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc$

$= (a + b + c) + 3$ ...(3)

Now $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$= 1 + 2.0 = 1$

$\Rightarrow a + b + c = 1$ (since a, b, c are real positive number)

Now from (3)

$a^3 + b^3 + c^3 = 1 + 3 = 4$.

Alternate:

$A^TA = I \Rightarrow |A^TA| = |I| \Rightarrow |A|^2 = 1$

$\Rightarrow (a^3 + b^3 + c^3 - 3abc)^2 = 1$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 1$ (since a, b, c are positive real number $\Rightarrow a^3 + b^3 + c^3 \geq 3abc$ from AM $\geq$ GM)

$\Rightarrow a^3 + b^3 + c^3 = 4$

4. $$\sum_{r=0}^{k}(-1)^r 2^{k-r}\, {}^nC_r\, {}^{n-r}C_{k-r} = \sum_{r=0}^{k}(-1)^r 2^{k-r}\, \frac{n!}{(n-r)!\, r!}\, \frac{(n-r)!}{(n-k)!\,(k-r)!}$$

$$= \sum_{r=0}^{k}(-1)^r 2^{k-r}\, \frac{n!}{(n-k)!\, k!}\, \frac{k!}{r!\,(k-r)!}$$

$${}^nC_k 2^k \sum \left(-\frac{1}{2}\right)^r {}^kC_r \quad {}^nC_k 2^k \left(1-\frac{1}{2}\right)^k \quad {}^nC$$

5. $\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \int_0^{\pi/4}\left[f(\cos 2x)\cos x + f\left(\cos 2\left(\frac{\pi}{2}-x\right)\right)\cos\left(\frac{\pi}{2}-x\right)\right]dx$

$= \int_0^{\pi/4}\left[f(\cos 2x)\cos x + f(-\cos 2x)\sin x\right]dx$

$= \int_0^{\pi/4} f(\cos 2x)\left[\cos x + \sin x\right]dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos\left(\frac{\pi}{4}-x\right)dx$

$= \sqrt{2}\int_0^{\pi/4} f\left(\cos 2\left(\frac{\pi}{4}-x\right)\right)\cos\left(\frac{\pi}{4}-\left(\frac{\pi}{4}-x\right)\right)dx$

$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$

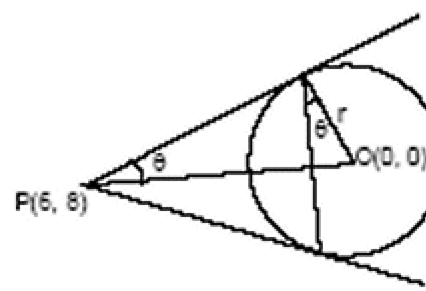
6. Let E_i : denotes the event that the student will pass the ith exam, i = 1, 2, 3
E: denotes the event that the student will qualify.

$P(E) = P(E_1) \times P\left(\frac{E_2}{E_1}\right) + P(E_1) \times P\left(\frac{E'_2}{E_1}\right) \times P\left(\frac{E_3}{E'_2}\right) + P(E'_1) \times P\left(\frac{E_2}{E'_1}\right) \times P\left(\frac{E_3}{E_2}\right)$

$= p^2 + p \times (1-p)\frac{p}{2} + (1-p) \times \frac{p}{2} \times p$

$\Rightarrow P(E) = \frac{2p^2 + p^2 - p^3 + p^2 - p^3}{2} = 2p^2 - p^3$

7. Since OP = 10, $\sin\theta = \frac{r}{10}$ where $\theta \in \left(0, \frac{\pi}{2}\right)$



$A = \frac{1}{2} \times 2r\cos\theta\,(10 - r\sin\theta)$
$= 10\sin\theta\cos\theta\,(10 - 10\sin^2\theta)$
$\Rightarrow A = 100\cos^2\theta\sin\theta\cos\theta$

$\frac{dA}{d\theta} = 100\left[\cos^4\theta - 3\sin^2\theta\cos^2\theta\right]$

$= 300\cos^4\theta\left(\frac{1}{\sqrt{3}} - \tan\theta\right)\left(\frac{1}{\sqrt{3}} + \tan\theta\right)$

$\Rightarrow$ A is maximum at $\theta = \frac{\pi}{6} \Rightarrow r = 10 \times \frac{1}{2} = 5$ units.

8 $a_1z + a_2z^2 + \ldots + a_nz^n = 1$
$\Rightarrow |a_1z + a_2z^2 + \ldots + a_nz^n| = 1$
$\Rightarrow |a_1z| + |a_2z^2| + \ldots + |a_nz^n| \ge 1$
$\Rightarrow 2[|z| + |z|^2 + \ldots + |z|^n] > 1$

$\Rightarrow \frac{2|z|\left(1-|z|^n\right)}{(1-|z|)} > 1$

as $|z| < \frac{1}{3}$, $|z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1} \Rightarrow |z| > \frac{1}{3}$
This is a contradiction.

P(E) = Probability that A will be hit

Then P(E) = $1-P(\overline{B}\cap\overline{C}) = 1-P(\overline{B}).P(\overline{C}) = 1-\frac{1}{2}.\frac{2}{3} = \frac{2}{3}$

$P\left(\frac{B\cap\overline{C}}{E}\right) = \frac{P(B).P(\overline{C})}{P(E)} = \frac{\frac{1}{2}.\frac{2}{3}}{\frac{2}{3}} = \frac{1}{2}$

10. L.H.D. at x = a

$f'_-(a) = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} = 0$ (given)

$f'_-(-a) = \lim_{h\to 0^-}\frac{f(-a+h)-f(-a)}{h} = \lim_{h\to 0^-}\frac{-f(-h+a)+f(a)}{h}$ (As f is odd)

$= \lim_{h\to 0^-}\frac{-f(2a+h-a)+f(a)}{h} = -f'_-(a) = 0.$

11. Let f (x) = sin (tan x) – x

$f'(x) = \cos(\tan x)\sec^2 x - 1 = \tan^2 x\cos(\tan x) + \cos(\tan x) - 1 > \tan^2 x\cos(\tan x) - \frac{\tan^2 x}{2}$

$\Rightarrow f'(x) > \tan^2 x(\cos(\tan x) - \cos\frac{\pi}{3}) > 0$ (since $0 \le \tan x \le 1 < \frac{\pi}{3}$)

$\Rightarrow$ f(x) is an increasing function $\forall\, x \in \left[0, \frac{\pi}{4}\right]$

As f(0) = 0 $\Rightarrow f(x) \ge 0\ \forall\, x \in \left[0, \frac{\pi}{4}\right]$

$\Rightarrow \sin(\tan x) \ge x.$

12. $2b = a + c$ ...(1)

$b^2 = \frac{2a^2c^2}{a^2+c^2} = \left(\frac{a+c}{2}\right)^2$

$\Rightarrow (a^2 + c^2)^2 + 2ac(a^2 + c^2) = 8a^2c^2$

$\Rightarrow (a^2 + c^2 + ac)^2 = 9a^2c^2$

$\Rightarrow a^2 + c^2 + ac = \pm 3ac$ ...(2)

$\Rightarrow a^2 + c^2 - 2ac = 0 \Rightarrow a = c \Rightarrow b = c$

or, $a^2 + c^2 = -4ac$

$\Rightarrow (a + c)^2 = -2ac$

$\Rightarrow 4b^2 = -2ac \Rightarrow b^2 = -\frac{ac}{2}$

$\Rightarrow a, b, -\frac{c}{2}$ are in G.P.

13. For unequal real roots

D > 0

$\Rightarrow (a - b)^2 - 4(1 - a - b) > 0$

$\Rightarrow b^2 + b(4 - 2a) + a^2 + 4a - 4 > 0$

For the above quadratic expression to be true $\forall\, b \in R$

Discriminant of its corresponding equation should be less than zero

i.e. $(4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow -32a + 32 < 0$

$\Rightarrow a > 1$.

$\Rightarrow m_1 m_2 m_3 = -k \Rightarrow m_3 = -\frac{k}{\alpha}$

$\Rightarrow \left(-\frac{k}{\alpha}\right)^3 - \frac{k}{\alpha}(2-h) + k = 0$

$\Rightarrow k^2 = \alpha^2 h - 2\alpha^2 + \alpha^3$

$\Rightarrow y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$

Comparing it with $y^2 = 4x$, we get $\alpha^2 = 4$ and $-2\alpha^2 + \alpha^3 = 0 \Rightarrow \alpha = 2$.

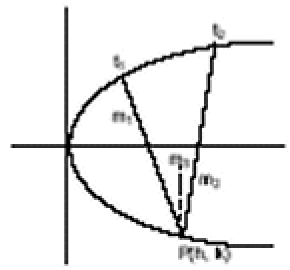
Alternate:

Since locus of P is a part of the parabola $\Rightarrow$ normals at any two points t_1 and t_2 meet at P

$\Rightarrow t_1 t_2 = 2$

$\Rightarrow (-m_1)(-m_2) = 2$

$\Rightarrow \alpha = 2$



15(i). From Lagrange's mean value theorem

$\frac{f(4)-f(0)}{4-0} = f'(a)$ for $a \in (0, 4)$ ...(1)

Also from Intermediate mean value theorem

$\frac{f(4)+f(0)}{2} = f(b)$ for $b \in (0, 4)$ ...(2)

From (1) and (2), we get

$\frac{(f(4))^2 - (f(0))^2}{8} = f'(a)f(b)$. for $a, b \in (0, 4)$

(ii). Replacing t by z^2, we get $\int_0^4 f(t)dt = \int_0^2 2zf(z^2)\,dz$

From Lagrange's mean value theorem

$\frac{\int_0^2 2zf(z^2)dz - \int_0^0 2zf(z^2)dz}{2-0} = 2\gamma f(\gamma^2)$ for $\gamma \in (0, 2)$

$\Rightarrow \int_0^2 2zf(z^2)dz = 2(2\gamma f(\gamma^2)) = 2\left(\frac{2\alpha f(\alpha^2) + 2\beta f(\beta^2)}{2}\right)$ (where $0 < \alpha < \gamma < \beta < 2$, using intermediate mean value theorem)

$\Rightarrow \int_0^4 f(t)dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]$ $\forall\ 0 < \alpha, \beta < 2$.

16(i). The equation of the plane is $\begin{vmatrix} x-2 & y-1 & z-0 \\ 5-2 & 0-1 & 1-0 \\ 4-2 & 1-1 & 1-0 \end{vmatrix} = 0 \Rightarrow (x-2)(-1) - (y-1)(1) + z(2) = 0$

$\Rightarrow x + y - 2z = 3$.

(ii). Let Q be (α, β, γ).

Equation of line PQ is $\frac{x-2}{1} = \frac{y-1}{1} = \frac{z-6}{-2}$

$\Rightarrow \frac{\frac{\alpha+2}{2}-2}{1} = \frac{\frac{\beta+1}{2}-1}{1} = \frac{\frac{\gamma+6}{2}-6}{-2}$

$$= \frac{1\left(\frac{\alpha+2}{2}-2\right)+1\left(\frac{\beta+1}{2}-1\right)-2\left(\frac{\gamma+6}{2}-6\right)}{1.1+1.1+(-2)(-2)} = 2 \qquad \left(\text{Since } 1\left(\frac{\alpha+2}{2}\right)+1\left(\frac{\beta+1}{2}\right)-2\left(\frac{\gamma+6}{2}\right)=3\right)$$

$\Rightarrow \alpha = 6,\ \beta = 5,\ \gamma = -2 \Rightarrow Q \equiv (6, 5, -2).$

17. $\frac{dP(x)}{dx} > P(x) \Rightarrow \frac{dP(x)}{dx} - P(x) > 0$

$\Rightarrow \frac{d}{dx}(P(x)e^{-x}) > 0$

$\Rightarrow P(x) \,.\, e^{-x}$ is an increasing function.

$\Rightarrow P(x)\, e^{-x} > P(1)\, e^{-1}\ \forall\, x \ge 1$

$\Rightarrow P(x)\, e^{-x} > 0\ \forall\, x > 1$ (since $P(1) = 0$)

$\Rightarrow P(x) > 0\ \forall\, x > 1.$

18. $I_n = \frac{n}{2} r^2 \sin\frac{2\pi}{n} \Rightarrow \frac{2I_n}{n} = \sin\frac{2\pi}{n}$ ....(1)

$O_n = nr^2 \tan\frac{\pi}{n}$ ....(2)

From (1) and (2), we get

$$\frac{I_n}{O_n} = \frac{1}{2}\,\frac{\sin\frac{2\pi}{n}}{\tan\frac{\pi}{n}} = \cos^2\frac{\pi}{n} = \frac{\cos\frac{2\pi}{n}+1}{2} = \frac{1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}}{2} \quad (\text{ using (1) })$$

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$$

19. $\vec{x} = \frac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|} = \frac{1}{2}\sec\frac{\alpha}{2}(\vec{u}+\vec{v})$, Similarly for vectors $\vec{y}$ and $\vec{z}$

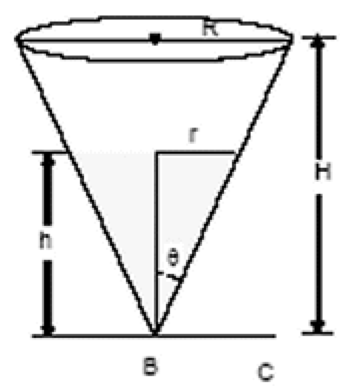
As $[(\vec{x}\times\vec{y})\ \ (\vec{y}\times\vec{z})\ \ (\vec{z}\times\vec{x})] = [\vec{x}\ \vec{y}\ \vec{z}]^2$

$= \frac{1}{64}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}+\vec{v}\ \ \vec{v}+\vec{w}\ \ \vec{w}+\vec{u}]^2$

$= \frac{4}{64}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}\ \ \vec{v}\ \ \vec{w}]^2$ As $[\vec{u}+\vec{v}\ \ \vec{v}+\vec{w}\ \ \vec{w}+\vec{u}] = 2[\vec{u}\ \vec{v}\ \vec{w}]$

$= \frac{1}{16}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}\ \ \vec{v}\ \ \vec{w}]^2$

20. Let the semi vertical angle of the cone be $\theta = \tan^{-1}\left(\frac{R}{H}\right)$



Let height of the liquid at time 't' be 'h' from the base BC and radius r.

Volume of liquid at time 't' = V = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \cot\theta$

S = Surface area in contact with air at time 't' = πr^2

Given that $-\frac{dV}{dt} \propto S$

$\Rightarrow -\frac{dV}{dt} = kS = k\pi r^2$

$$\Rightarrow \frac{\cot\theta}{3}\pi 3r^2 \frac{dr}{dt} = -k\pi r^2$$

$$\Rightarrow \cot\theta \int_R^0 dr = -k\int_0^T dt \quad \text{(where T is the required time)}$$

$$\Rightarrow \frac{H}{R} R = kT \Rightarrow T = \frac{H}{k}$$

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