MATHEMATICAL SCIENCES

REAL ANALYSIS

(i) A *'set' its 'elements'* and *an element belongs to a set'* are basic undefined concepts in Mathematics.

(ii) A set, is generally specified by a well-defined collection of objects. Sets are usually denoted by capital letters and their elements are denoted by small letters.

(iii) A set is said to be *finite* or *infinite* according as it has a finite or infinite number of elements respectively.

(iv) A set having no element is called the *null set* or the *empty set* or the *void set* and is denoted by ϕ. A set having a single element is called the *singleton set*.

(v) A set A is called the *subset* of another set B if every element of A is also an element of B and we write $A \subseteq B$. If there exists at least one element in B that is not in A, we say that A is a *proper subset* of B and we write $A \subset B$. Note that ϕ is a subset of every set and each set is a subset of itself.

(vi) Two sets A and B are said to be equal written as $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

(vii) The cardinal number of a set A, denoted by $n(A)$, is defined as the number of elements in A.

(viii) Two sets A and B are said to be *equivalent* if $n(A) = n(B)$.

(ix) The family of all subsets of any set A is called the *power set* of A and is denoted by $P(A)$. If A has n elements, then the power set of A contains 2^n elements.

(x) The union of two sets A and B, written as $A \cup B$, is defined as

$$A \cup B = \{x/x \in A \text{ or } x \in B\}$$

For n sets $A_1, A_2, \ldots, A_n$, we define

$$\bigcup_{k=1}^{n} A_k = \{x / x \in A_k, \text{ for some } k, k = 1, 2, \ldots n\}$$

(xi) The intersection of two sets A and B, written as $A \cap B$ is defined as

$$A \cap B = \{x/x \in A \text{ or } x \in B\}.$$

For n sets $A_1, A_2, \ldots, A_n$, we define

$$\bigcup_{k=1}^{n} A_k = \{x / x \in A_k, \text{ for every } k, k = 1, 2, \ldots n\}$$

(xii) The difference of two sets A and B, written as $A - B$ is defined as

$$A - B = \{x/x \in A \text{ or } x \notin B\}$$

Note that $A - B \subseteq A$ and $B - A \subseteq B$.

(xiii) The symmetric difference of two sets A and B, written as $A \Delta B$ is defined as

$$A \Delta B = (A - B) \cup (B - A).$$

(xiv) The complement of a set A, written as A', is defined as

$$A' = \{x/x \notin A\}$$

(xv) *De Morgan's law:*

(a) $(A \cup B) = A' \cap B'$

(b) $(A \cap B) = A' \cup B'$

For n sets $A_1, A_2, \ldots, A_n$, the above laws become:

(a') $\left(\bigcup_{k=1}^{n} A_k'\right) = \bigcap_{k=1}^{n} A_k'$

(b') $\left(\bigcap_{k=1}^{n} A_k'\right) = n\bigcup_{k=1}^{n} A_k'$

(xvi) **Properties of cardinal Numbers of sets**

(1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(2) $n(A \cup B) = n(A, B) + n(B - A) + n(A \cap B)$.

(3) $n(A \cup B \cup C) = n(A) + n(B) +$

$n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

(xvii) **The cartesian product of two sets**, written as $A \times B$, is defined as

$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$.

(xviii) **Relations**

(a) A relation R between two sets A and B is a subset of the cartesian product $A \times B$, *i.e.*, $R \subseteq A \times B$. We write $R = \{(a, b) | a R b\}$.

(b) The **inverse relation of R**, written as R^{-1}, is a subset of $B \times A$, defined by $R^{-1} = \{(b, a) | (a, b) \in R\}$

Note that $(R^{-1})^{-1} = R$.

(c) If $R \subseteq A \times B$ and $S \subseteq B \times C$, then the **composite relation** $(SoR) \subseteq A \times C$ is defined by

$SoR = \{(a, c) / \exists$ a point $b \in B$ s.t. $(a, b) \in R$ and $(b, c) \in S\}$.

Note that $SoR \neq RoS$ and that $(SoR)^{-1} = R^{-1} o S^{-1}$.

(d) A relation R defined on a set A is said to be a **binary relation** on A if $R \subseteq A \times A$.

(e) A binary relation R defined on a set A is said to be:

1. **reflexive** if $aRa \ \forall \ a \in A$.
2. **symmetric** if $aRb \Rightarrow bRa \ \forall \ a, b \in A$.
3. **anti-symmetric** if aRb and $bRa \Rightarrow a = b, \forall \ a, b \in A$.
4. **transitive** if aRb and $bRc \Rightarrow aRc$, $\forall \ a, b, c \in A$.

(f) A binary relation R defined on a set A is said to be an **equivalence relation** on A if and only if it is reflexive, symmetric and transitive.

(xix) **Binary operations on a set**

A binary operation (denoted by * or O) on a set A is a mapping which associates each ordered pair $(a, b) \in A \times A$ with a unique member $a * b \in A^{*}$, *i.e.*,

$* : A \times A \to A$

$: (a, b) \to a * b$

Note that if a binary operation * is well-defined on A, we say that A is *closed* w.r.t. *.

(a) A binary operation * on a set A is said to be:

(i) **commutative** if $a * b = b * a$, $\forall \ a, b \in A$.

(ii) **associative** if $(a * b) * c = a * (b * c)$ $\forall \ a, b, c \in A$.

(iii) **distributive** over another binary operation o if

$a * (boc) = (a * b) o (a * c)$, and

$(boc) * a = (b * a), o (c * a)$, $\forall \ a, b, c \in A$.

(iv) **Satisfying cancellation laws**

If $\forall \ a, b, c \in A$,

$b * a = c * a \Rightarrow b = c$ and

$a * b = a * c \Rightarrow b = c$

(b) If there exists an element $e \in A$ such that $\forall \ a \in A$,

$a * e = e * a = a$

then e is called the identity element in A w.r.t. *.

(c) If there exist an element $b \in A$ s.t.

$a * b = b * a = e$

then b is called the **inverse** of an element $a \in A$.

FUNCTION

1. A function f is a relation in which no two different ordered pairs have the same first element.

(a) The *domain* of a function f is the subject of $X : D(f) = \{x : x$ is the first element of atleast one of the pairs (x, y) of $f\}$.

(b) The *range* of a function f is the subset of $Y : R(f) = \{y : y$ is the second element of atleast one of the pairs (x, y) of $f\}$.

2. Let $f : X \to Y$. If $R(f) = Y$, we say that f is an *onto function* or *surjective*. If for all $x_1, x_2 \in X$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, f is called a *one-one function* or *injective*. A function which is both injective and surjective is called a *bijective* function.

3. If A and B have m and n elements respectively, then

(i) The no. of relations between A and B is 2^{mn}.

(ii) The no. of mappings or functions from A and B is n^m.

(iii) The no. of bijective mappings from A to A is $m!$

4. A function of the type $y = a_0x^n + a_1x^{n-1} + \dots a_n$, where $a_0, a_1, \dots, a_n$ are real constants, n is a non-negative integer and $a_0 \neq 0$, is called a *polynomial function*.

5. The function $y = |x|$ is called an *absolute value function* or *modulus function*. For this function $D(f) = R$ and $R(f) = R^+$.

6. The function $y = [x]$, where $[x]$ denotes the *greatest integer less than or equal to x*, is called the *greatest integer function* or the *step function*. Here $D(f) = R$ and $R(f) = Z$.

7. The function $y = sign(x)$ is called the *signum function*. Here $D(f) = R$ and $R(f) = \{-1, 0, 1\}$.

8. **Algebra of functions:**

For two real-valued functions f and g, let $D = D(f) \cap D(g)$. Then the define:

(i) **Sum of Difference:** $(f \pm g)x = f(x) \pm g(x)$, $\forall x \in D$.

(ii) **Scalar multiplication:** $(cf)x = cf(x)$, $\forall f \in D(f)$ and $c \in R$.

(iii) **Product:** $(fg)x = f(x)g(x)$, $\forall x \in D$.

(iv) **Quotient:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $\forall x \in D - \{x; g(x) = 0\}$.

9. **Composite function:** If $f: X \to Y$, $g: Y \to Z$ then $g \circ f: X \to Z$ defined by $(gof)(x) = g(f(x))$, $\forall x \in X$

Note that $g \circ f \neq f \circ g$.

10. **Inverse of a function:** Let $f: X \to Y$. If f is bijective i.e., *one-one* and *onto*, then $f^{-1}: Y \to X$ such that $f^{-1}(y) = x$, $\forall y \in Y$ and $x \in X$. Note that $D(f^{-1}) = R(f)$ and $R(f^{-1}) = D(f)$.

Properties:

1. f^{-1} exists $\Leftrightarrow f$ is *one-one* and *onto*.
2. $(f^{-1})^{-1} = f$.
3. $(f^{-1} \circ f)$ is the identity function on X [illegible]

4. The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

11. **Exponential functions:** The function $y = a^x$, where $a > 0$ and $a \neq 1$; is called the exponential function. Its perticular case $y = e^x$ is very useful.

Here $D(f) = R$ and $R(f) = R^+ - \{0\}$.

If $0 < a < 1$, $y = a^x$ is decreasing and if $a > 0$, $y = a^x$ is increasing.

12. **Logarithmic functions:** The logarithmic function is defined as $y = log_a x$, where $x > 0$, $a > 0$, $a \neq 1$.

For $x \leq 0$, this function is not defined. If $a = e$, we write $log_e x = ln x$.

$D(f) = R^+ - \{0\}$, $R(f) = R$

If $0 < a < 1$, $y = log_a x$ is decreasing and if $a > 0$, $y = log_a x$ is increasing.

1. THEOREMS ON LIMITS OF FUNCTIONS:

(i) The limit of a function, if it exists, is always unique.

(ii) The limit of a constant is that constant itself.

(iii) $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$, c is constant.

(iv) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$.

This result is true for a sum of difference of any finite number of functions.

(v) $\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x).\lim_{x \to a} g(x)$.

(vi) $\lim_{x \to a} [f(x)/g(x)] = \lim_{x \to a} f(x)/\lim_{x \to a} g(x)$, provided $\lim_{x \to a} g(x) \neq 0$.

2. ONE-SIDED LIMITS

(i) **Right hand limit:** $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$.

(ii) **Left hand limit:** $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$.

If these one-sided limits exists, are finite and equal then

$\lim f(x) = \lim f(x) = \lim g(x)$ [illegible]

3. SOME-WELL KNOWN LIMITS

(i) $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} \cos x = 1$.

(ii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$ and $\lim_{x\to 0}\left(\frac{x}{\sin x}\right) = 1$.

(iii) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1$ and $\lim_{x\to 0}\left(\frac{x}{\tan x}\right) = 1$.

(iv) $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right) = 0$.

(v) $\lim_{x\to(\pi/2)^-} (\tan x) = \infty$ and $\lim_{x\to(\pi/2)^-} (\tan x) = \infty$.

(vi) $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$.

(vii) $\lim_{x\to\infty} (\log x) = \infty$ and $\lim_{x\to 0^+} (\log x) = -\infty$.

(viii) $\lim_{x\to 0}(1+x)^{1/x} = e$ and $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

4. CONTINUITY

A function $y = f(x)$ is said to be *continuous* at a point $x = a$ if $f(a)$ exists, $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$; otherwise it is said to be *discontinuous* at $x = a$.

5. DERIVATIVE

If $y = f(x)$ is a function of x, then

$$\frac{dy}{dx} = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\},$$

provided this limit exists. Geometrically speaking *the derivative of a function at a point equals the slope of the tangent line to the curve at that point.*

(i) If the tangent is *parallel* to x-axis $\Rightarrow \frac{dy}{dx} = 0$.

(ii) If the tangent is *perpendicular* to x-axis $\Rightarrow \frac{dy}{dx} = 0$.

6. DIFFERENTIABILITY AND CONTINUITY

If a function $y = f(x)$ is differentiable at a point, then it is necessarily continuous at that point. The converse is not true, in general.

7. SOME THEOREMS ON DIFFERENTIATION:

(i) **Power Formula:** $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, $n \in Q$.

(ii) $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$, c being a constant.

(iii) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$, u, v being functions of x.

This theorem holds for a sum or difference of any finite number of functions.

(iv) $\frac{d}{dx}(u.v) = \left(\frac{du}{dx}\right).v + u\left(\frac{dv}{dx}\right)$.

(v) $\frac{d}{dx}\left(\frac{u}{v}\right) = \left\{\frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{2}\right\}$, $v \ne 0$.

8. ROLLE'S THEOREM

Let $y = f(x)$ be a function defined on a closed interval $[a, b]$ such that

(i) $f(x)$ is continuous on $[a, b]$,

(ii) $f(x)$ is differentiable on (a, b),

(iii) $f(a) \ne f(b)$,

then there exists *at least* one real number c, $a < c < b$, such that $f'(c) = 0$.

9. LAGRANGE'S MEAN VALUE THEOREM

Let $y = f(x)$ be a function of x defined on $[a, b]$ such that

(i) $f(x)$ is continuous on $[a, b]$,

(ii) $f(x)$ is differentiable on (a, b),

(iii) $f(a) \ne f(b)$,

then, there exists *at least* one real number c, $a < c < b$, such that $\frac{f(b) - f(a)}{(b-a)} = f'(c)$.

10. TEST FOR INCREASE AND DECREASE OF A FUNCTION

Let $y = f(x)$ be a differentiable function defined on (a, b). Then, f is

(i) increasing on (a, b) if $f'(x) \geq 0 \;\forall\; x \in (a, b)$.

(ii) decreasing on (a, b) if $f'(x) \leq 0 \;\forall\; x \in (a, b)$.

11. A NECESSARY CONDITION FOR MAXIMUM OR MIMIMUM

If a differentiable function $y = f(x)$ has a maximum or a minimum at a point x then,

$$\frac{dy}{dx} = 0.$$

The converse is not true, in general.

12. SUFFICIENT CONDITIONS FOR A MAX. OR MIN.

Let $y = f(x)$ be a continuous function defined on an interval $[a, b]$ containing a critical point say $x = x_1$. Let $f(x)$ be differentiable at all points of $[a, b]$ except, possibly, at $x = x_1$ itself. In passing through the point $x = x_1$ from left to right, if the derivative changes sign from

(i) +ive to −ive $\Rightarrow f(x)$ has a **maximum** at $x = x_1$.

(ii) −ive to +ive $\Rightarrow f(x)$ has **minimum** at $x = x_1$.

Remark: The above test is known as the **first derivative test** for maxima and minima

13. THE SECOND DERIVATIVE TEST FOR MAXIMA AND MINIMA

Let $y = f(x)$ be a successively differentiable function and $x = x_1$ be a critical point of $f(x)$, i.e. $f'(x_1) = 0$. Then

(i) $f''(x_1) < 0 \Rightarrow f(x)$ has a **maximum** at $x = x_1$

(ii) $f''(x_1) > 0 \Rightarrow f(x)$ has a **minimum** at $x = x_1$

Remark: If $f''(x_1) = 0$, we say that the *second derivative test fails*. In general, if first $(n-1)^{th}$ order derivatives vanish but the n^{th} order derivative is non-zero, then

Case I. If n is *even* (i.e. n = 4, 6, ...) then

$f^{(n)}(x_1) < 0 \Rightarrow f(x)$ has a **maximum** at $x = x_1$.

$f^{(n)}(x_1) > 0 \Rightarrow f(x)$ has **minimum** at $x = x_1$.

Case II. If n is *odd* (i.e., n = 3, 5, ...) the $f(x)$ has **neither a maximum nor a minimum**.

In such cases,

$f^{(n)}(x_1) > 0 \Rightarrow f(x)$ is *increasing*,

$f^{(n)}(x_1) < 0 \Rightarrow f(x)$ is *decreasing*,

in a small neighbourhood of the point $x = x_1$.

14. MAX. AND MIN. OF TRIGONOMETRIC FUNCTIONS

Since trigonometric functions are periodic, it is sufficient to investigate the function for maxima and minima on the interval $[-\pi, \pi]$ or $[0, 2\pi]$.

15. L'HOSPITAL'S RULE

Suppose we have to find $\lim_{x \to a} \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are two functions of x such that $f(a) = 0$ and $g(a) = 0$, and then $f(x)/g(x)$ assumes the indeterminate from $\frac{0}{0}$ when $x = a$. However, we can determine the limiting values of such expressions if such value exist.

If $f(x)$ and $g(x)$ are successively differentiable functions of x in the neighbourhood of the point $x = a$, then $\lim_{x \to a} \{f(x)/g(x)\} = \lim_{x \to a} [f'(x)/g'(x)]$ provided the limit on $R.H.S.$ exists.

If the value of $R.H.S.$ is not immediately obvious, and if $f'(a) = g'(a) = 0$, then $\lim_{x \to a} \{f(x)/g(x)\} = \lim_{x \to a} \left\{\frac{f''(x)}{g''(x)}\right\}$, provided the limit on $R.H.S.$ exists, and so on.

Derivatives of Trigonometric and Inverse Trigonometric functions:

(i) $\frac{d}{dx}(\sin x) = \cos x$.

(ii) $\frac{d}{dx}(\cos x) = -\sin x$.

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$.

(iv) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

(v) $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(vi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(vii) $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{(1-x^2)}}$.

(viii) $\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{1}{\sqrt{(1-x^2)}}$.

(ix) $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{(1+x^2)}$.

(x) $\frac{d}{dx}\left(\cot^{-1}x\right) = \frac{1}{(1+x^2)}$.

(xi) $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{(x^2-1)}}$.

(xii) $\frac{d}{dx}\left(\text{cosec}^{-1}x\right) = \frac{1}{x\sqrt{(x^2-1)}}$.

Derivatives of Logarithmic and Exponential Functions:

(i) $\frac{d}{dx}(\log x) = \frac{1}{x}$.

(ii) $\frac{d}{dx}(\log_a x) = \frac{1}{x(\log a)}$.

(iii) $\frac{d}{dx}(a^x) = a^x(\log a)$.

(iv) $\frac{d}{dx}(e^x) = e^x$.

Higher Derivatives:

(i) $D^n x^n = n!$ and $D^{n+1}x^n = 0,\ n \in N$.

(ii) $D^n e^{mx} = m^n e^{mx}$.

(iii) $D^n a^x = a^x(\log a)^n$

(iv) $D^n \sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

1. Standard Formulae of Integration

(i) $\int dx = x + c$.

(ii) $\int x^n dx = \frac{x^{n+1}}{(n+1)} + c$, $n \in Q,\ n \neq -1$

(iii) $\int \frac{1}{x} dx = \log x + c$.

(iv) $\int e^x dx = e^x + c$.

(v) $\int a^x dx = \frac{a^x}{(\log a)} + c,\ (a > 0,\ b \neq 1)$.

(vi) $\int \sin x\, dx = -\cos x + c$.

(vii) $\int \cos x\, dx = \sin x + c$.

(viii) $\int \sec^2 x\, dx = \tan x + c$.

(ix) $\int \text{cosec}^2 x\, dx = -\cot x + c$.

(x) $\int \sec x \tan x\, dx = \sec x + c$.

(xi) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$.

(xii) $\int \frac{1}{\sqrt{(1-x^2)}} dx = \sin^{-1}x + c$.

(xiii) $\int \frac{1}{(1-x^2)} dx = \tan^{-1}x + c$.

(xiv) $\int \tan x\, dx = -\log\cos x + c = \log\sec x + c$.

(xv) $\int \cot x\, dx = -\log\sin x + c$.

(xvi) $\int \text{cosec}\, x\, dx = \log\tan\left(\frac{x}{2}\right) + c$.

$= \log\ (\text{cosec}\, x - \cot x) + c$.

(xvii) $\int \sec x\, dx = \log\tan\left(\frac{x}{2}+\frac{\pi}{2}\right) + c$.

$= \log(\sec x + \tan x) + c$

(xviii) $\int \frac{1}{\sqrt{(a^2-x^2)}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$.

(xix) $\int \frac{1}{\sqrt{(a^2-x^2)}} dx = \log\left[x+\sqrt{x^2+a^2}\right] + c$.

(xx) $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \log\left[x+\sqrt{x^2+a^2}\right] + c$.

(xxi) $\int \frac{1}{\sqrt{(x^2-a^2)}} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

(xxii) $\int \frac{1}{(x^2-a^2)} dx = \frac{1}{2a}\log\left(\frac{x-a}{a-x}\right) + c$.

(xxiii) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left(\frac{x+a}{a-x}\right) + c$.

(xxiv) $\int (x^2+a^2) dx$ [illegible] $\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + c$

(xxv) $\int \sqrt{(x^2+a^2)} dx = \frac{1}{2}x\sqrt{a^2+a^2}$ [illegible]

$+\frac{1}{2}a^2 \log\left|x+\sqrt{(x^2+a^2)}\right|+c$

(xvii) $\int\sqrt{(x^2+a^2)}\,dx = \left(\frac{1}{2}\right)x\sqrt{(x^2+a^2)}$

$+\left(\frac{1}{2}\right)a^2\log\left[x+\sqrt{(x^2+a^2)}\right]+c.$

Fundamental Rules of Integration

(i) $\int[f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx.$

(ii) $\int cf(x)dx = c\int f(x)dx$, c being a constant.

Integration by Parts:

(i) $\int f(x) \cdot g(x)dx$

$= \left\{\int g(x)dx\right\} - \int\left[f'(x).\int g(x)dx\right]dx + C$

Properties of Definite Integrals:

(i) $\int_a^b f(x)dx = -\int_b^a f(t)dt = \int_a^b f(u)du = \ldots$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a < c < b.$

(iv) $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$

(v) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$

(vi) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x). \end{cases}$

Integral as the Limit of a Sum:

$\int_a^b f(x)dx = \lim_{n\to\infty} h\left[f(a)+f(a+h)+\ldots+f(a+\overline{n-1}h)\right],$

where $(b-a) = nh$.

Area by Integration:

(i) The area A bounded by the curve $y = f(x)$, the x-axis, and the ordinates at $x = a$ and $x = b$ is given by

$A = \int_a^b y\,dx = \int_a^b f(x)dx$

(ii) The area A bounded by the curve $x = f(y)$, the y-axis, and the abscissae at $y = a$ and $y = b$, is given by

$A = \int_a^b f(y)dy = \int_a^b x\,dy$

(iii) The area A between two curves $y = f_1(x)$ and $y = f_2(x)$ is given by

$A = \int_a^b \left[f_1(x) - f_2(x)\right]dx.$

DIFFERENTIAL EQUATION

(a) Order of a D.E.:

The order of a *D.E.* is the order of the highest derivative that occurs in it.

(b) Degree of a D.E.:

The **degree** of a *D.E.* is the highest power on the highest derivative appearing in it, provided the derivative or derivatives do not occur under the radicals and fractions.

(c) General Solution and Particular Solutions:

The General Solution or the complete primitive or the complete integral of a *D.E.* is a solution which contains as many arbitary constant as is the order of the *D.E.* Solutions obtained from the general solution by assigning particular values to the arbitary constants of the general solution, are known as **particular solution.**

(d) D.E's 1st order and 1st degree:

(i) **V-S form:** A *D.E.* is said to be of **variables-separable form** if it can be written as

$f_1(x)dx \pm f_2(y)dy = 0$

A direct Integration gives the desired Solution.

(ii) **Reducible to V-S form:** The general form of such equation is $\frac{dy}{dx} = f(ax+by+c)$. To solve it, put $f(ax+by+c) = u$, the given *D.E.* then reduces to V-S form in u and x.

(iii) **Homogeneous form:** $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$, where

f_1 and f_2 are homogeneous functions of x and y of the *same* degree. To solve, it, put $y = vx$, where v is an unknown function of x.

(iv) **Linear form:** $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x or constants. Its $I.F. = e^{\int P dx}$ and its general solution is

$y.e^{\int P dx} = \int \{e^{\int P dx} Q\} dx + c$

(v) $e^{\log m} = m$.

(e) Equations of the form $\frac{d^2y}{dx^2} = f(x)$

To solve such equations, integrate it twice.

VECTORS

Vector Algebra:

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors and m, n are scalars, then the following are satisfied:-

(i) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

(ii) $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$.

(iii) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$.

(iv) $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$.

(v) $m(n\vec{a})=(mn)\vec{a}$.

(vi) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$.

(vii) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$.

Section Formula: The position vector of a point C which divides the line joining the points A and B with position vectors $\vec{a}$ and $\vec{b}$ respectively relative to some origin 0, in a given ratio $m : n$, is given by

$$\vec{c} = \frac{n\vec{a} + m\vec{b}}{m + n}$$

(i) If C is the mid point of A and B, then the position vector of C is given by

$$\vec{c} = \frac{1}{2}(\vec{a} + \vec{b})$$

(ii) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the vertices of a triangle ABC, then the position vector of the centroid G of ΔABC relative to the origin O is given by

$$\vec{OG} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$$

Three given points A, B, C with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ relative to some origin 0, are **collinear** if $\vec{AC} = \lambda\vec{AB}$, where λ is a constant.

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then $|\vec{r}|=\sqrt{(x^2+y^2+z^2)}$.

Three vector $\vec{a}$, $\vec{b}$ and $\vec{c}$ are said to be:

(i) **Linearly independent** if $m\vec{a}+n\vec{b}+p\vec{c}=\vec{0}$ implies $m = 0$, $n = 0$ and $p = 0$.

(ii) **Linearly dependent** if $m\vec{a}+n\vec{b}+p\vec{c}=\vec{0}$ implies that m, n and p are not all zero. Note that three vectors in 3D are linearly independent $\Leftrightarrow$ if they are non-coplanar.

Dot Product:

(i) If $\vec{a}$. $\vec{b}$ are two vectors and θ is the angle between them, then

$\vec{a} . \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

(ii) $\vec{a} . \vec{b}$ is a scalar quantity.

(iii) Scalar Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} . \vec{b}}{|\vec{b}|}$

Scalar Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} . \vec{b}}{|\vec{a}|}$

(iv) $\vec{a} . \vec{b} = \vec{b} . \vec{a}$

(v) $\vec{a} . (\vec{b}+\vec{c}) = \vec{a} . \vec{b} + \vec{a} . \vec{c}$

(vi) $m (\vec{a} . \vec{b}) = (m\vec{a}) . \vec{b} = \vec{a} . (m\vec{b})$

(vii) $\vec{a}^2 = \vec{a} . \vec{a} = |\vec{a}|^2$

(viii) $\vec{a} . \vec{b} = 0 \Rightarrow$ Either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a}$ is *perpendicular* to $\vec{b}$.

Hence for *non-zero* vectors $\vec{a}$ and $\vec{b}$,

$\vec{a} . \vec{b} = 0 \Leftrightarrow$ and $a \perp b$.

(ix) **Dot Products of $\hat{i}$, $\hat{j}$ and $\hat{k}$**

$(\hat{i} . \hat{i})=1$, $(\hat{j} . \hat{j})=1$, $(\hat{k} . \hat{k})=1$

and $(\hat{i} . \hat{j}) = 0$, $(\hat{j} . \hat{k}) = 0$, $(\hat{k} . \hat{i}) = 0$

(x) **Dot Products in terms of components:**

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Cross Product:

(i) If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them, then

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$,

where $\hat{n}$ is a *unit vector perpendicular* to both $\vec{a}$ and $\vec{b}$.

(ii) $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$

(iii) The direction of $\vec{a} \times \vec{b}$ is *perpendicular* to both $\vec{a}$ and $\vec{b}$.

(iv) If $\vec{a}$ and $\vec{b}$ are *parallel* (or *collinear*),

$\vec{a} \times \vec{b} = \vec{0}$

(v) The cross product is not commutative, *i.e.*,

$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

In fact $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

(vi) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(vii) $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

(viii) **Cross Products of $\hat{i}$, $\hat{j}$ and $\hat{k}$**

$(\hat{i} \times \hat{i}) = \vec{0}, (\hat{j} \times \hat{j}) = \vec{0}, (\hat{k} \times \hat{k}) = \vec{0}$

$(\hat{i} \times \hat{j}) = \hat{k}, (\hat{j} \times \hat{k}) = \hat{i}, (\hat{k} \times \hat{i}) = \hat{j}$

$(\hat{j} \times \hat{i}) = -\hat{k}, (\hat{k} \times \hat{j}) = -\hat{i}, (\hat{i} \times \hat{k}) = -\hat{j}$.

(ix) **Cross product in terms of components**

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(x) **Geometrical Meaning of Cross Product:**

The scalar area of the parallelogram with vectors $\vec{a}$ and $\vec{b}$ forming its adjacent sides is given by $A = |\vec{a} \times \vec{b}|$

(xi) **Vector Area of a triangle:**

The vector Area $\vec{\Delta}$ of a triangle A B C is given by

$\vec{\Delta} = \frac{1}{2}[(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})]$,

where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of *A*, *B* and *C* respectively relative to some origin O.

(xii) **Vector area of a parallelogram:**

The vector area of a parallelogram is equal to half the vector area of the parallelogram determined by its diagonals, *i.e.*,

$\vec{a} \times \vec{b} = \frac{1}{2}[(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})]$.

Multiple products:

For three vectors **a**, *b* and **c**.

(i) $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the **scalar triple product** (*s.t.p.*) of $\vec{a}$, $\vec{b}$ and $\vec{c}$. It is a scalar quantity. In short $\vec{a} \cdot (\vec{b} \times \vec{c})$ is written as $[\vec{a}\,\vec{b}\,\vec{c}]$.

(ii) $\vec{a} \times (\vec{b} \times \vec{c})$ is called the **vector triple product** (*v.t.p.*) of $\vec{a}$, $\vec{b}$ and $\vec{c}$. It is a vector quantity.

(iii) The volume V of the parallelopiped with vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, forming its adjacent sides is given by

$V = [\vec{a}\,\vec{b}\,\vec{c}]$.

(iv) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

(v) In a *s.t.p.*, if any two vectors are equal, the product vanishes, i.e.,

$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0, (\vec{b} \times \vec{c}) = 0$,

$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0, \vec{c} \cdot (\vec{a} \times \vec{c}) = 0$ etc.

(vi) **s.t.p. in terms of components**

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(vii) Three non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are **coplaner**, if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

(viii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$.

(ix) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(x) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix}$

(xi) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\,\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{d}$

Reciprocal System of Vectors

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-zero and non-coplaner vectors then the three vectors $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ defined by

$$\vec{a}' = \frac{(\vec{b} \times \vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{b}' = \frac{(\vec{c} \times \vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]};\ \text{and}\ \vec{c}' = \frac{(\vec{a} \times \vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

are called the **reciprocal system** of vectors for the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Properties:

(*i*) $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are also non-coplaner.

(*ii*) $\vec{a} . \vec{a}' = 1$, $\vec{b} . \vec{b}' = 1$ and $\vec{c} . \vec{c}' = 1$. Hence the nomenclature is justified.

(*iii*) $\vec{a}.\vec{b}' = \vec{a}.\vec{c}' = \vec{b}.\vec{c}' = \vec{b}.\vec{a}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$.

(*iv*) $[\vec{a}'\,\vec{b}'\,\vec{c}'] = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$.

PROBABILITY:

1. (*i*) The probability of an event A, is given by

$$P(A) = \frac{\text{No. of outcomes favourable to A}}{\text{No. of all equally likely events}} = \frac{n(A)}{n(S)}.$$

(*ii*) $0 \le P(A) \le 1$.

(*iii*) The probabilities of a *certain event* and an *impossible event* are 1 and zero respectively.

(*iv*) Two events are said to be *mutually exclusive* if they cannot occur simultaneously in a single outcome. If A and B are two mutually exclusive events, then $P(A \cap B) = 0$.

(*v*) Two events A and B are said to be *independent* if the occurrence or non-occurrence of any one of them does not affect the possibility of occurrence or non-occurrence of the other.

(*vi*) Two events A and B are said to be *dependent* if the occurrence of A affects the occurrence of B.

(*vii*) If A and B are any two events of a sample space S, then

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

If A and B are *mutually exclusive* events,

$P(A \text{ or } B) = P(A) + P(B)$.

(*viii*) If A and B are independent events,

$P(A \text{ and } B) = P(A) \times P(B)$.

(*ix*) $P(A) = 1 - P(\bar{A})$.

(*x*) $P(A \cap B) = P(B/A) \cdot P(A) = P(A/B) \cdot P(B)$.

(*xi*) For any three events A, B, C of a sample space S,

$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

2. Probability Distribution

The probability distribution of a random variable X tells us as to how the total probability 1 is distributed over the several values of the random variable X.

(*i*) **Mean or expected value of a random variable**

$$\mu = \Sigma p_i x_i.$$

(*ii*) **Variance of a random variable**

$$\alpha x^2 = \Sigma p_i (x_i - \mu)^2.$$

If μ is a fraction, we have

$$\sigma x^2 = \Sigma p_i x_i - \mu^2.$$

(iii) **S.D. of a random variable**

$$\sigma = +\sqrt{\alpha x^2}.$$

3. The Binomial or Bernoulli Distribution

(*a*) $P(X = r) = {}^nC_r p^r q^{n-r}$, where $r = 0, 1, ..., n$;

n = no. of trials.

p = probability of success.

q = probability of failure.

(*b*) The probability of **atleast r** success is given by

$$P(X \le r) = 1 - \sum_{x=r+1}^{n} {}^nC_x p^x q^{n-x}.$$

(*c*) The probability of **atmost r** success is given by

$$P(X \le r) = 1 - \sum_{x=r+1}^{n} {}^nC_x p^x q^{n-x}.$$

(*d*) For a binomial distribution,

Mean : $\mu = np$

Variance : $\sigma^2 = npq$

S.D. : $\sigma = \sqrt{npq}$

(*e*) Recurrence formula

$$P(r+1) = \left(\frac{n-r}{r+1}\right) \cdot \frac{p}{q} \cdot P(r),$$

$r = 0, 1, 2, \ldots, (n-1)$.

NUMERICAL METHOD

1. Location theorem:

Let $y = f(x)$ be a real-valued, continuous function defined on $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one real root in (a, b).

2. Numerical methods of solving an equation:

(*i*) **Bolzano's bisection method:** If a real root of the equation $f(x) = 0$ lies in the interval $[a, b]$, then the *first approximation* to the desired root is given by

$$x_1 = \tfrac{1}{2}(a + b).$$

Again, if the desired root lies in $[a, x_1]$, then the *second approximation* is given by

$$x_2 = \tfrac{1}{2}(a + x_1).$$

Continue the process until the root is obtained to the desired degree of accuracy.

(*ii*) **Regula Falsi method:** If a real root of the equation $f(x) = 0$ lies in the interval $[a, b]$, then the *first approximation* to the desired root is given by

$$x_1 = a - \left[\frac{(b-a)f(a)}{f(b)-f(a)}\right]$$

Again, if the desired root lies between a and x_1 then *second approximation* is given by

$$x_2 = a - \left[\frac{(x_1-a)f(a)}{f(x_1)-f(a)}\right].$$

The process is repeated till the root is obtained to the desired degree of accuracy.

(*iii*) **Newton-Raphson method:** The *n*th approximation to a real root of the equation $f(x) = 0$ is given by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

provided $f'(x_{n-1}) \neq 0$, $n = 1, 2, 3, \ldots$

5. Approximations of functions:

Working rule:

(*i*) Write down the power series of the given function.

(*ii*) Write down T_r and T_{r+1}

(*iii*) Find T_{r+1}/T_r, $r = 1, 2, 3, \ldots$

(*iv*) Assuming $T_0 = 0$, calculate the value of T_r for the given value of x. Hence find T_{r+1} by using $T_{r+1} = RT_r$.

(*v*) Apply the formula $S_{r+1} = S_r + T_{r+1}$, where $S_0 = 0$ and $S_1 = T_1$.

(*vi*) Repeat steps (iv) and (v) for $r = 1, 2, 3, \ldots$, and continue till T_{r+1} become zero upto the desired degree of accuracy.

(vii) The final value of S_{r+1} gives the desired approximate value of $f(x)$ for a given value of x.

Complex Analysis:

(*i*) If $z = x + iy$, then $x = Re(z)$ and $y = Im(z)$.

The complex number z is *purely real* or *purely imaginary* according as $Im(z) = 0$, or $Re(z) = 0$ respectively.

(*ii*) **The geometrical representation of complex numbers as points in the plane is known as the Argand diagram.**

(*iii*) The polar form of a complex number is $z = r(\cos\theta + i\sin\theta)$ where $r > 0$.

(*iv*) The *modulus* of z, written as $|z|$, is given by

$$|z| = \sqrt{(x^2 + y^2)}.$$

Note that $|z|^2 = |z^2|$.

(*v*) The angle θ, called the *argument* of z is given by $\arg(z) = \tan^{-1}(y/x)$ and *considering the quadrant in which z lies, where* $z \neq 0$.

(*vi*) There is no order relation in *C*. i.e., given any two complex numbers z_1 and z_2, we cannot say whether $z_1 > z_2$ or $z_1 < z_2$.

(vii) **Operations in C.**

(a) **Addition:** $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

(b) **Multiplication:** $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

(c) **Division:**

$$\left(\frac{z_1}{z_2}\right) = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right),$$

(d) **Equality:** $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$.

(e) $\bar{z} = x - iy$ is called the **conjugate of Z.**

(f) The multiplicative inverse of z is given by $z^{-1} = \frac{\bar{z}}{|z|^2}$.

Properties of conjugacy:

(1) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(2) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(3) $z\bar{z} = |z|^2$

(viii) **Cube roots of unity:**

The three cube roots of 1 are 1, ω and ω^2 where $\omega = \frac{-1 + i\sqrt{3}}{2}$.

Properties:

(1) $\omega^3 = 1$.

(2) $1 + \omega + \omega^2 = 0$.

(ix) **De Moivre's Formula:**

If θ is measured in radians and $n \in Q$, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(x) **Square roots of a complex number:**

Every non-zero complex number $z = x + iy$ has exactly two square roots, i.e.,

$\sqrt{x - iy} = \pm(a + ib)$

where $a^2 - b^2 = x$,

$2ab = y$

and $a^2 + b^2 = \sqrt{x^2 + y^2}$.

LINEAR ALGEBRA

1. An equation of the form $ax^2 + bx + c = 0$, is called a quadratic equation, where $a, b, c \in R$.

2. Methods of Solving quadratic equations

(i) By completing the perfect square.

(ii) By factorisation, if convenient.

(iii) By applying the quadratic formula:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

3. Nature of the roots:

The roots of the equation:

$ax^2 + bx + c = 0$ are $\alpha = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}$ and

$\beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$

Note that Sum of the roots: $\alpha + \beta = -b/a$.

and Product of the roots: $\alpha\beta = c/a$.

(i) If $(b^2 - 4ac) = 0 \Rightarrow$ Both α, β are real and **equal**.

(ii) If $(b^2 - 4ac) > 0 \Rightarrow$ Both α, β are real and **distinct**.

(iii) If $(b^2 - 4ac) < 0 \Rightarrow$ Both α, β are complex and distinct.

(iv) If $a, b, c \in Q$, then if $(b^2 - 4ac) \geq 0$ and it is a perfect Square then both α and β are **rationals**. However, if $(b^2 - 4ac) > 0$ and is not a **perfect square** then both α and β are **irrational**.

(v) If $p + iq$ is a root of a quadratic equation, then $p - iq$ is also a root of that equation, provided a, b, c are all real.

4. Formation of a quadratic equation:

If the roots of a quadratic equation are given, say r_1 and r_2 then the required quadratic equation is

$$(x - r_1)(x - r_2) = 0,$$

or, $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$

5. The following results are very useful for solving quadratic inequations.

(i) $x^2 \leq k^2 \Leftrightarrow -k \leq x \leq +k$.

(ii) $x^2 \geq k^2 \Leftrightarrow -k \geq x$ or $x \geq k$.