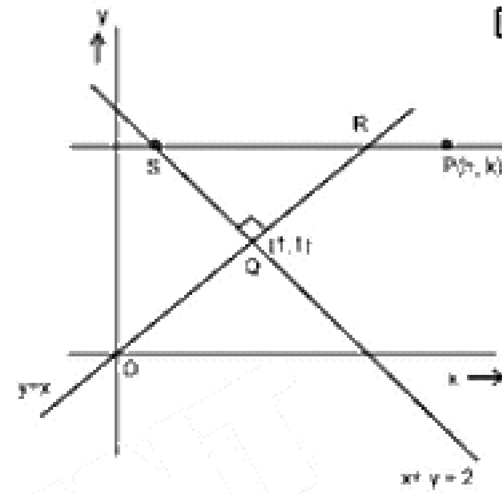
Solutions to IIT JEE-2005 Mains Paper (Memory based)

Mathematics

Q.1 A line passes through the point P(h, k) is parallel to the x- axis. It forms a triangle with the lines $y = x$ & $x + y = 2$ of area $4h^2$ then find the locus of P. **[2]**



Sol. On solving the lines we get S & R points

S : (2 – k, k) , R : (k, k)

Two lines intersect at Q (1, 1)

Area of the (ΔQRS) = $\frac{1}{2} \times 2\,(k-1)^2$ $\quad$ [$\because$ QS = QR = $\sqrt{2}$ SR]

$4h^2 = [k-1]^2$

$\Rightarrow$ locus is the pair of straight lines $\boxed{4x^2 = (y-1)^2}$ **Ans.**

Q.2 A cricket player in his career plays n match and scores total no. of $\frac{(n+1)(2^{n+1}-n-2)}{4}$ runs. **[2]**

Sol. If he scores $k.\,2^{n-k+1}$ runs in k^{th} match, where $1 \le k \le n$. Find n.

Let S_n be the total scores in his career plays n matches.

$$S_n = \sum_{k=1}^{n} k.2^{n+1-k} = 2^{n+1} \sum_{k=1}^{n} k.2^{-k}$$

$$= 2^{n+1}\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} - \frac{n}{2^{n+1}}\right]\frac{1}{1-\frac{1}{2}}$$

$$= 2^{n+2}\left[1-\frac{1}{2^n}-\frac{n}{2^{n+1}}\right] = 2^{n+2} - 4 - 2n \;= 2^{n+2} - 2n - 4$$

But $S_n = \frac{n-1}{4}(2^{n+1} - 2 - n)$ (as given)

so $\frac{n+1}{4}(2^{n+1} - 2 - n) = 2(2^{n+1} - n - 2)$

$n + 1 = 8$ $\qquad$ so $\boxed{n = 7}$ **Ans.**

Q.3 Ramesh goes to office either by car, scooter, bus or train probability of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$ respectively and probability that he is reaching office late if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$ respectively. Find the probability that he has travelled by car, if he reaches office in time. **[2]**

Sol. Let A, B, C, D be the events when Ramesh is going by car, scooter , bus or train respectively.

$P(A) = \frac{1}{7}$, $P(B) = \frac{3}{7}$, $P(C) = \frac{2}{7}$, $P(D) = \frac{1}{7}$

Let $\bar{x}$ be the event when Ramesh reaching the office in time.

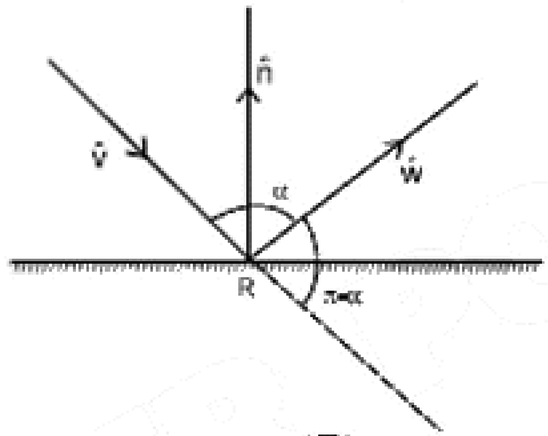
$P \quad = \frac{7}{9}$, $P\left(\frac{\bar{x}}{B}\right) = \frac{8}{9}$, $P\left(\frac{\bar{x}}{C}\right) = \frac{5}{9}$, $P\left(\frac{\bar{x}}{D}\right) = \frac{8}{9}$

$$P\left(\frac{A}{\bar{x}}\right) = \frac{P\left(\frac{\bar{x}}{A}\right).P(A)}{P(\bar{x})} = \frac{\frac{7}{9}\times\frac{1}{7}}{\frac{1}{7}\times\frac{7}{9}+\frac{3}{7}\times\frac{8}{9}+\frac{2}{7}\times\frac{5}{9}+\frac{8}{9}\times\frac{1}{7}} = \frac{1}{7} \textbf{ Ans.}$$

Q.4 A light ray incident along the unit vector $\hat{v}$ and the unit vector along the normal to reflecting surface at the point P is $\hat{n}$ outwards. If reflecting ray is along the unit vector $\hat{w}$, find $\hat{w}$ in terms of $\hat{v}$ and $\hat{n}$. **[2]**

Sol. $\hat{n}$ is bisector of the angle between the incident and reflected ray and can be represented as

$$\hat{n} = \frac{\hat{w}-\hat{v}}{|\hat{w}-\hat{v}|}$$



where $|\hat{w}-\hat{v}| = \sqrt{1+1-2(\hat{w}.\hat{v})}$ [where $\pi-\alpha$ is angle between $\hat{w}$ and $\hat{v}$]

$= |2\cos\alpha/2| \quad$ where $\alpha \in (0, \pi)$

$= -2(\hat{n}.\hat{v})$

$\Rightarrow \boxed{\hat{w} = \hat{v} - 2(\hat{n}.\hat{v})\,\hat{n}}$ **Ans.**

Q.5 Find the equation of the plane at distance of $\frac{1}{\sqrt{6}}$ from the point (2, 1, – 1) and containing the line $2x - y + z - 3 = 0 = 3x + y + z - 5$. **[2]**

Sol. $(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$

$(3\lambda + 2)x + (\lambda - 1)y + (\lambda + 1)z - 5\lambda - 3 = 0$

$$\left|\frac{6\lambda+4+\lambda-1-\lambda-1-5\lambda-3}{\sqrt{(3\lambda+2)^2+(\lambda-1)^2+(\lambda+1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$6(\lambda-1)^2 = 11\lambda^2 + 12\lambda + 6$

$\lambda = 0, -\frac{24}{5}$

So the planes are $2x - y + z - 3 = 0$ and $62x + 29y + 19z - 105 = 0$ **Ans.**

Q.6 For any two distinct real numbers x_1 & x_2, $y = f(x)$ is satisfying the condition, $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$. Find the equation of the tangent at the point (1, 2) to the curve $y = f(x)$. **[2]**

Sol. Given : $|f(x_1) - f(x_2)| < |x_1 - x_2|^2$

Let $x_2 = 1$ & $x_1 = 1 + h$

$|f(1+h) - f(1)| < |h|^2$

so $\lim\limits_{h\to 0}\left|\frac{f(1+h)-f(1)}{h}\right| < \lim\limits_{h\to 0}|h|$

$|f'(1)| < \epsilon$; where is a very small positive quantity.

so $f'(1) = 0$

Hence the tangent to $y = f(x)$ at (1, 2) is $y = 2$. **Ans.**

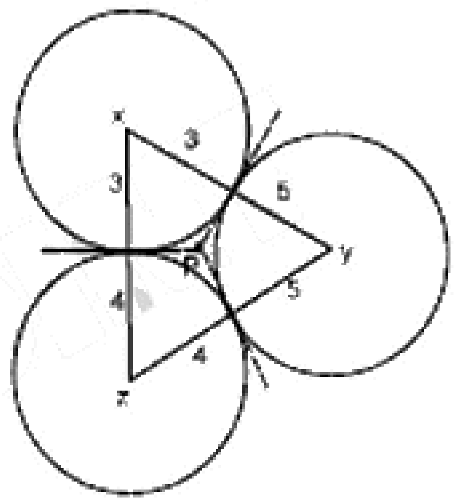
Q.7 3, 4, 5 are radii of three circles touch each other externally if P is the point of intersection of tangents of these circles at their points of contact, find the distances of P from the points of contact. **[2]**

Sol. Let x, y, z be the centre of the three circles clearly the point P is the in- centre of the Δxyz and

$r = \quad = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$; where Δ & S have their usual meaning

$2s = 7 + 8 + 9 \Rightarrow s = 12$

$r = \sqrt{\frac{5.4.3}{12}} = \sqrt{5}$ unit. **Ans.**



Q.8 Evaluate : $\int_0^{\pi} e^{|\cos x|}[2\sin(1/2\cos x) + 3\cos(1/2\cos x)]\sin x\,dx$ **[2]**

$\frac{\Delta}{s}$

Sol. Let $\frac{1}{2}\cos x = t$

$\cos x = 2t$ so $-\sin x\,dx = 2dt$

so $I = \int_{-1/2}^{1/2} e^{|2t|}[\,2\sin(t) + 3\cos(t)]\,2dt$

$= 2\int_{-1/2}^{1/2} 2.e^{|2t|}\sin t\,dt + 6\int_{-1/2}^{1/2} e^{|2t|}\cos t\,dt$ (Using $\int_{-a}^{a} f(x)dx = 0$; if f(x) is odd function

$= 2\int_0^a f(x)\,dx$; if f(x) is even function)

$= 0 + 12\int_0^{1/2} e^{2t}\cos t\,dt$

$= \frac{12}{\sqrt{5}}[e^{2t}\cos(t - \tan^{-1}\frac{1}{2})]_0^{\frac{1}{2}}$

$= \frac{12}{\sqrt{5}}[\,e\cos(\frac{1}{2} - \tan^{-1}\frac{1}{2}) - \cos(\tan^{-1}\frac{1}{2})]$

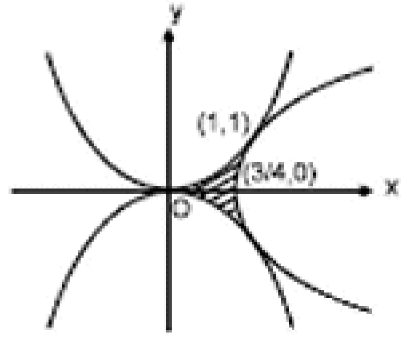
$= \frac{12}{\sqrt{5}}[e\cos(\frac{1}{2} - \tan^{-1}\frac{1}{2}) - \frac{2}{\sqrt{5}}]$ **Ans.**

Q.9 Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ & $y^2 = 4x - 3$. **[4]**

Sol. On solving the curves $x^2 = y$ and $y^2 = 4x - 3$

We get their point of contact (1,1) and the area bounded is symmetric about x- axis.

so that the area bounded is



$$= 2\left[\int_0^{3/4} x^2 + \int_{3/4}^{1} (x^2 - \sqrt{4x-3})\, dx\right]$$

$$= \frac{1}{3} \text{ sq. units.} \quad \textbf{Ans.}$$

Q.10 Find the equation of the common tangent in 1st quadrnt to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Also find the length of the intercept of the tangent between the coordinates axes. **[4]**

Sol. Let the equations of tangents to the given circle and the ellipse respectively.

$y = mx + 4\sqrt{1+m^2}$ ....(1)

and $y = mx + \sqrt{25m^2 + 4}$ ....(2)

since (1) & (2) are coincident lines, so

$4\sqrt{1+m^2} = \sqrt{25m^2+4}$

$16(1 + m^2) = 25\, m^2 + 4$

$m = \pm\frac{2}{\sqrt{3}}$

$m < 0$ because common tangent in 1st quadrant

so $m = -\frac{2}{\sqrt{3}}$

so the equation of the common tangent is ; (from (1))

$y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$

It meets the coordinates axes at A $(2\sqrt{7}, 0)$ and $(0, 4\sqrt{\frac{7}{3}})$ so length of the intercept of the tangent between the coordinate axes is $\frac{14}{\sqrt{3}}$. **Ans.**

Q.11 A tangent is drawn from a point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to circle $x^2 + y^2 = 9$ find the locus of mid-point of chord of contact. **[4]**

Sol. Let any point $P(3\sec\theta, 2\tan\theta)$ on the hyperbola

chord of contact of the circle $x^2 + y^2 = 9$ with respect to the point $(3\sec\theta, 2\tan\theta)$ is

$(3\sec\theta) \cdot x + (2\tan\theta)\, y = 9$ .....(1)

Let (h, k) be the mid point of the chord of contact.

equation of chord in mid-point form is $xh + yk = h^2 + k^2$ ....(2)

By (1) and (2)

$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2+k^2}$$

$\sec\theta =$, $\tan\theta = \dfrac{9k}{2(h^2+k^2)}$

as $\sec^2\theta - \tan^2\theta = 1$, so

$$\frac{81h^2}{9(h^2+k^2)^2} - \frac{81k^2}{4(h^2+k^2)^2} = 1$$

so the required locus is $\boxed{(x^2+y^2)^2 = 9x^2 - \frac{81}{4}y^2}$ **Ans.**

Q.12 A square having one vertex as $2 + \sqrt{3}\,i$ is circumscribed on the circle $|z - 1| = \sqrt{2}$. Then find the other vertices of square. **[4]**

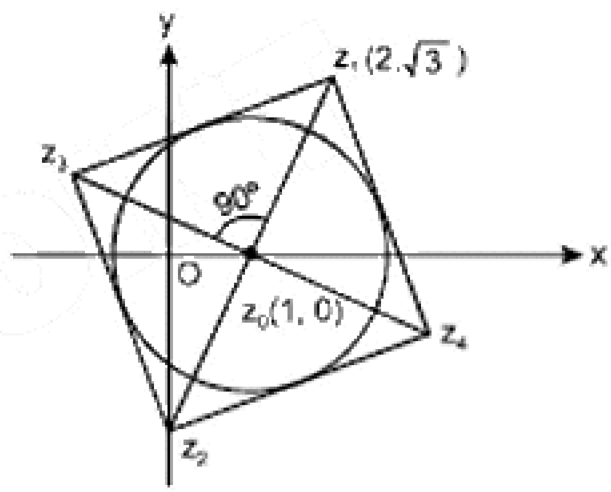
Sol. Centre of circle is (1, 0) and is also the mid point of diagonals of square

$z_3 - z_0 = (z_1 - z_0)\, e^{i\pi/2}$

$= (2 + \sqrt{3}\,i - 1)\, i$

$z_3 = i - \sqrt{3} + 1$

$= (1 - \sqrt{3} + i)$



Since z_0 is mid point of z_1 & z_2 as well as z_3 & z_4 also

so $z_4 = 2z_0 - z_3 = (1 + \sqrt{3}) - i$

and $z_2 = 2z_0 - z_1 = 2 - (2 + \sqrt{3}\,i) = -\sqrt{3}\,i$

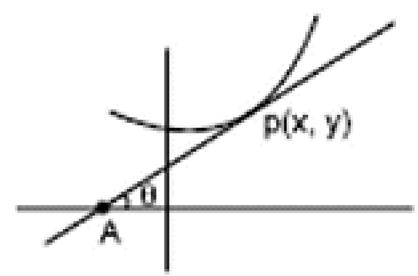
so other vertices are $(0, -\sqrt{3})$, $(1 - \sqrt{3}, 1)$, $(1 + \sqrt{3}$, $\frac{9h}{3(h^2+k^2)}$ Ans.

Q.13 Find all the curves y = f(x), tangents at any point on which are drawn such that the segment of tangent intercepted between the contact point and x-axis is of unit length. **[4]**

Sol. Given : | AP | = 1

Equation of tangent at P(x, y) point is

$$Y - y = \frac{dy}{dx}(X - x)$$



Putting X = 0 & Y = 0 respectively we get pts A & B and length of the segment AB is

| AP | is length of tangent, so

$$|AP| = \left|\frac{y}{y'}\sqrt{1+(y')^2}\right| = 1 \qquad \text{where } y' = \left(\frac{dy}{dx}\right)_P$$

$$y^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = \left(\frac{dy}{dx}\right)^2$$

$$\int \frac{\sqrt{1-y^2}}{y}\,dy = \pm\int dx$$

on integrating we get,

$$\boxed{\sqrt{1-y^2} + \ln\left|\frac{1-\sqrt{1-y^2}}{y}\right| = \pm x + c}$$ **Ans.**

Q.14 If two functions 'f' and 'g' satisfying given conditions for $\forall$ x, y $\in$ R. f(x – y) = f(x) g(y) – f(y) . g(x) and g(x – y) = g(x) . g(y) + f(x) f(y). If right hand deviative at x = 0 exists for f(x) then find the deviative of g(x) at x = 0. **[4]**

Sol. Given $f(x-y) = f(x) \cdot g(y) - f(y) \cdot g(x)$

Put $y = x$ and you get $f(0) = 0$

put $y = 0$ and you get $g(0) = 1$

R.H.D. of f(x) : $f'(0^+) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ (h $\in$ R^+ and tending to zero)

$$= \lim_{h\to 0} \frac{f(0)g(-h) - g(0)f(-h) - f(0)}{h} = \lim_{h\to 0} \frac{-f(-h)}{h} = f'(0^-) = \text{L.H.D.} \quad .....(1)$$

& L.H.D. of f(x) : $f'(0^-) = \lim\limits_{h\to 0} \dfrac{f(0-h)-f(0)}{-h} = -f'(0^+) = -\text{R.H.D.}$ ....(2)

Hence from (1) and (2) $f'(0) = 0$

Put y = x in given condition $g(x-y) = g(x) \cdot g(y) + f(x) \cdot f(y)$

$\Rightarrow g(0) = g^2(x) + f^2(x)$

on diff. w.r.t.x we get $\Rightarrow g^2(x) + f^2(x) = 1 \quad \Rightarrow g^2(x) = 1 - f^2(x)$

$\Rightarrow 2g(x) \cdot g'(x) + 2f(x) f'(x) = 0$

$g'(0) = 0$

[Note : 'g' is differentiable at zero because 'f' is diff. at 0 & $g^2(x) = 1 - f^2(x)$]

Q.15 Find the range of values of t for which $2 \sin t = \dfrac{1-2x+5x^2}{3x^2-2x-1}$; $t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **[4]**

Sol. Given : $2 \sin t = \dfrac{1-2x+5x^2}{3x^2-2x-1}$

First we have to check the range of R.H.S. $\dfrac{1-2x+5x^2}{3x^2-2x-1} = y$ (say)

$(3x^2 - 2x - 1)\, y - (1 - 2x + 5x^2) = 0$

$(3y - 5)\, x^2 - 2x\, (y - 1) - (y + 1) = 0$

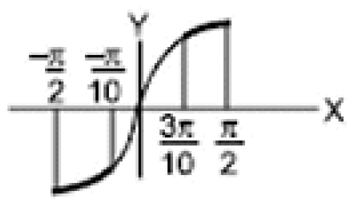
So $D \geq 0$

since $x \in R - \{1, -\dfrac{1}{3}\}$

$y^2 - y - 1 \geq 0$

or $y \geq \dfrac{1+\sqrt{5}}{2}$ or $y \leq \dfrac{1-\sqrt{5}}{2}$ $\left(\because \sin\dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4} \text{ \& } \sin\dfrac{3\pi}{10} = \dfrac{\sqrt{5}+1}{4}\right)$

or $\sin t \geq \dfrac{1+\sqrt{5}}{4}$ or $\sin t \leq \dfrac{1-\sqrt{5}}{4}$



$\Rightarrow \sin t \geq \sin\dfrac{3\pi}{10}$ or $\sin t \leq \sin\left(-\dfrac{\pi}{10}\right)$

$\Rightarrow$ $\boxed{\dfrac{\pi}{2} \geq t \geq \dfrac{3\pi}{10} \quad \text{or} \quad -\dfrac{\pi}{2} \leq t \leq -\dfrac{\pi}{10}}$ **Ans.**

Q.16 If P(x) be the cubic polynomial satisfying p (–1) = 10, p(1) = – 6 and p(x) has maximum at x = –1 and p′(x) has minima at x = 1. Find the points of local maxima and minima, also find the distance between these two points. **[4]**

Sol. Let $P(x) = ax^3 + bx^2 + cx + d$

$P(-1) = -a + b - c + d = 10$ ....(1)

& $P(1) = a + b + c + d = -6$ ....(2)

$P'(x) = 3ax^2 + 2bx + c$

since $P'(-1) = 3a - 2b + c = 0$ ....(3)

and $P''(x) = 6ax + 2b$

$P''(1) = 6a + 2b = 0$

solving (1), (2), (3) & (4) for a, b, c, d

$a = 1, b = -3, c = -9, d = 5$

so $P(x) = x^3 - 3x^2 - 9x + 5$

Now $P'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

+   –   +
–1   3

So P(x) has minima at x = 3 & maxima at x = – 1

so the point of local max. (3, – 22)

so the point minima (–1, 10)

distance = $\sqrt{16+(32)^2} = \sqrt{16+1024} = \sqrt{1040} = 4\sqrt{65}$ unit **Ans.**

Q.17 f (x) be a quadratic polynomial , a, b,c are three distinct real numbers, such that :

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$$

V is the point where f (x) attains maximum. A & B are the points on f(x) such that f (x) cuts x –axis at A in the first quadrant and chord AB subtends right angle at V. Find the area bounded by curve y = f (x) and chord AB. **[6]**

Sol. As given $4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a$.

$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b$ and

$4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$

here it can be considered that the equation :

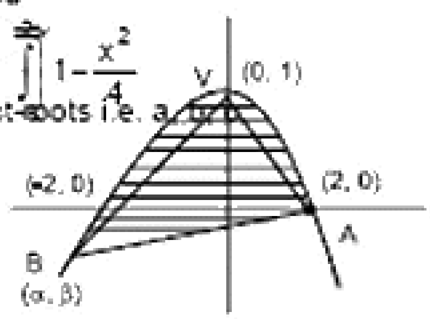
$4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$ has three distinct roots i.e. a, b, c

which is possible if and only if

$4 f(-1) = 3 \Rightarrow f(-1) = 3/4$

and $4 f(1) = 3$ $f(1) = 3/4$

$f(2) = 0$



so the function $f(x) = \frac{-x^2}{4} + 1$

Let A be at (2 , 0). let B be (α, β)

so $\beta = -\frac{\alpha^2}{4} + 1$ & ....(1)

Since BV ⊥ AV so

$-\frac{1}{2}\left(\frac{\beta-1}{\alpha}\right) = -1 \Rightarrow \beta = 2\alpha + 1$ ....(2)

from (1) & (2)

(α, β) = (– 8, – 15) = point B

so the required area = $\int \left(1 - \frac{x^2}{4} - \frac{(-6+3x)}{2}\right) dx$

area = $\frac{125}{3}$ sq. units **Ans.**

Q.18 If 'f' is differentiable and 'g' is a double differentiable function such that $f(x) \in [-1, 1]$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$ then prove that there exist some $c \in (-3, 3)$ such that $g(c)\, g''(c) < 0$ **[6]**

Sol. We have to prove that $g(x).\, g''(x) < 0$ for some $c \in (-3, 3)$ means both can not be +ve or –ve for all $x \in (-3, 3)$ simultaneously. First we assume both are +ve for all $x \in (-3, 3)$.

$f(x) \in [-1, 1]$ and $f^2(0) = 9$

$\Rightarrow g(0) \in [\quad , 3]$

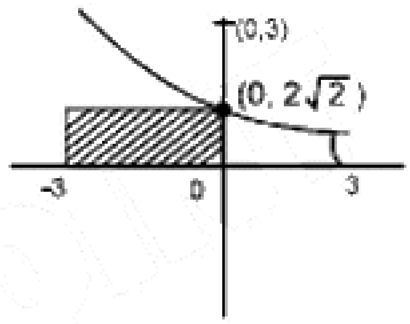
given $f'(x) = g(x)$

$$\int_{-3}^{x} f'(x)dx = \int_{-3}^{x} g(x)dx$$

$$f(x) = \int_{-3}^{x} g(x)dx + f(-3) \quad [f(-3) \text{ can take min}^m \text{ value } -1]$$

since $g''(x) > 0 \Rightarrow$ curve is opening upwards

for any g (x) satisfying the conditions, why ?

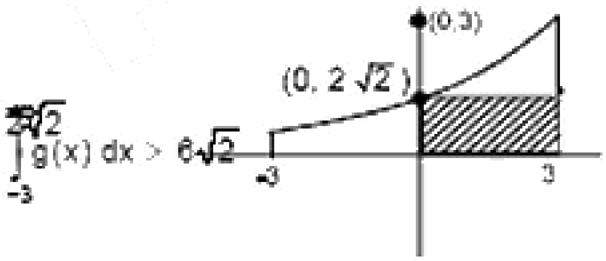


you can understand by following cases –

Case I : If g(x) is decreasing.

$$\int_{-3}^{x} g(x)dx > 3 \times 2\sqrt{2} \text{ (area of rectangle)}$$

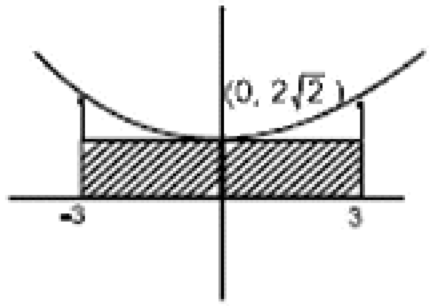
Case II : If g (x) is increasing



$$\int_{-3}^{x} g(x)dx > 6\sqrt{2} \text{ (area of rectangle)}$$

Case II : If g(x) takes min^m at x = 0

$$\int_{-3}^{g(x)} g(x)dx > 6 \times 2\sqrt{2} > 6\sqrt{2} \text{ (area of rectangle)}$$



$f(x) > 6\sqrt{2} - 1$

but f (x) can not be greater than one so that their shows contradiction means assumed condition can not be true any how.

$\Rightarrow$ g (x) and g'' (x) can not be both +ve for simultaneously all $x \in (-3\ 3)$

for some $c \in (-3, 3)$ $\qquad g(c).\, g''(c) < 0$

similarly you can prove that both can not be –ve simultaneously.