BOARD ANSWER PAPER: MARCH 2013
MATHEMATICS AND STATISTICS

SECTION – I

Q.1. (A) Select and write the correct answer from the given alternatives in each of the following:

i. (C)

$\cot x = -\sqrt{3}$

$\therefore \tan x = -\frac{1}{\sqrt{3}}$

$= -\tan\frac{\pi}{6}$

$= \tan\left(\pi - \frac{\pi}{6}\right)$

$= \tan\frac{5\pi}{6}$

$\therefore x = \tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ [2]

ii. (D)

Since, the given vectors are coplanar.

$\therefore \begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -p & 0 \end{vmatrix} = 0$

$\therefore$ p = 2 [2]

iii. (C)

The line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

if $c = \pm\sqrt{a^2m^2 - b^2}$

Here, $a^2 = 16$, $b^2 = 9$, m = 1

$\therefore c = \pm\sqrt{16-9}$

$= \pm\sqrt{7}$ [2]

(B) Attempt any THREE of the following:

i. a. Let p : A triangle is equilateral,
q : A triangle is equiangular.

$\therefore$ the symbolic form of the given statement is $p \leftrightarrow q$. [1]

b. Let p : Price increases, q : Demand falls

$\therefore$ the symbolic form of the given statement is $p \wedge q$. [1]

ii. $|A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 2(3) - 4(-2) = 14 \neq 0$

$\therefore$ A^{-1} exists.

$A_{11} = (-1)^{1+1} M_{11} = 3$

$A_{12} = (-1)^{1+2} M_{12} = -(4) = -4$

$A_{21} = (-1)^{2+1} M_{21} = -(-2) = 2$

$A_{22} = (-1)^{2+2} M_{22} = 2$

Hence, matrix of the co-factors is

$\left[A_{ij}\right]_{2\times2} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$ [1]

Now, adj A $= \left[A_{ij}\right]^{T}_{2\times2} = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{|A|}(\text{adj } A) = \dfrac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$ [1]

iii. Given equation is $3x^2 - 10xy - 8y^2 = 0$

$\therefore$ $3x^2 - 12xy + 2xy - 8y^2 = 0$

$\therefore$ $3x(x - 4y) + 2y(x - 4y) = 0$

$\therefore$ $(x - 4y)(3x + 2y) = 0$ [1]

$\therefore$ the separate equations of the lines are $x - 4y = 0$ and $3x + 2y = 0$. [1]

iv. Given equation of the circle is $x^2 + y^2 = 100$.

Here, $a^2 = 100$

The director circle of the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$. [1]

$\therefore$ the equation of the director circle is $x^2 + y^2 = 2\,(100)$ i.e., $x^2 + y^2 = 200$. [1]

v. $4\cos^2 x = 1$

$\therefore$ $\cos^2 x = \dfrac{1}{4}$

$\therefore$ $\cos^2 x = \left(\cos\dfrac{\pi}{3}\right)^2 = \cos^2\left(\dfrac{\pi}{3}\right)$ [1]

Since, $\cos^2\theta = \cos^2\alpha$ implies $\theta = n\pi \pm \alpha$, $n \in Z$.

$\therefore$ the required general solution is $x = n\pi \pm \dfrac{\pi}{3}$, where $n \in Z$. [1]

Q.2. (A) Attempt any TWO of the following:

i. L.H.S. $= p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

$\equiv [\sim p \wedge (\sim q \vee p)] \vee [q \wedge (\sim q \vee p)]$ .... (Distributive law) [1]

$\equiv [(\sim p \wedge \sim q) \vee (\sim p \wedge p)] \vee [(q \wedge \sim q) \vee (q \wedge p)]$ .... (Distributive law)

$\equiv [(\sim p \wedge \sim q) \vee F] \vee [F \vee (q \wedge p)]$ .... (Complement law) [1]

$\equiv (\sim p \wedge \sim q) \vee (q \wedge p)$ .... (Identity law)

$\equiv (q \wedge p) \vee (\sim p \wedge \sim q)$ .... (Commutative law)

$\equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ .... (Commutative law) [1]

$\equiv$ R.H.S.

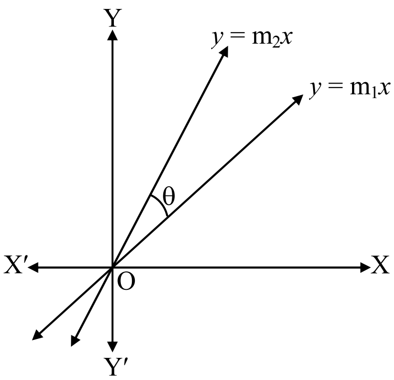
ii. The given combined equation of lines is $ax^2 + 2hxy + by^2 = 0$
Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$

∴ $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$, $b \neq 0$

If $\frac{a}{b} = -1$, then $m_1 \cdot m_2 = -1$.

∴ lines are perpendicular. [1]

So we assume that $\frac{a}{b} \neq -1$



Now, $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1.m_2$

$$= \left(-\frac{2h}{b}\right)^2 - \frac{4a}{b} = \frac{4h^2}{b^2} - \frac{4a}{b}$$

∴ $(m_1 - m_2)^2 = \frac{4h^2 - 4ab}{b^2} = \frac{4(h^2 - ab)}{b^2}$

Taking square root on both the sides, we get

$m_1 - m_2 = \pm \frac{2\sqrt{h^2 - ab}}{b}$ [1]

Let θ be the acute angle between the lines.

∴ $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1.m_2}\right| = \left|\frac{\pm\frac{2\sqrt{h^2 - ab}}{b}}{1 + \frac{a}{b}}\right|$, $\frac{a}{b} \neq -1$

∴ $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$, $a + b \neq 0$ [1]

iii. Equation of the parabola is $y^2 = 8x$ .... (i)
and equation of the line is $x + 2y + 8 = 0$
i.e., $x = -2y - 8$ .... (ii)
To find the point of intersection of the line and parabola,
Putting (ii) in (i), we get
$y^2 = 8(-2y - 8)$
∴ $y^2 + 16y + 64 = 0$
∴ $(y + 8)^2 = 0$ [1]
This quadratic equation in y has equal roots, each being -8.
Hence, the given line touches the parabola at the point whose y – coordinate is -8. [1]
When $y = -8$, $x = -2(-8) - 8 = 8$
∴ the point of contact is $(8, -8)$. [1]

(B) Attempt any TWO of the following:

i. Let the three numbers be x, y, z.

According to the given conditions,

$x + y + z = 9$

$y + 3z = 16$

$x - 2y + z = 6$ [1]

Matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \\ 6 \end{bmatrix}$$ [1]

Applying $R_3 \to R_3 - R_1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \\ -3 \end{bmatrix}$$

Applying $R_3 \to R_3 + 3 R_2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \\ 45 \end{bmatrix}$$

By equality of matrices,

$x + y + z = 9$ .... (i)

$y + 3z = 16$ .... (ii)

$9z = 45$ .... (iii) [1]

From (iii), z = 5

Putting z = 5 in (ii), we get

$y + 3 \times 5 = 16$

∴ $y = 16 - 15 = 1$

Putting $y = 1$, z = 5 in (i) we gt

$x + 1 + 5 = 9$

∴ $x + 6 = 9$

∴ $x = 9 - 6 = 3$

∴ $x = 3, y = 1, z = 5$

Hence, 3, 1 and 5 are the required numbers. [1]

ii. $\cos x + \sin x = 1$

Dividing throughout by $\sqrt{2}$, we get

$$\frac{1}{\sqrt{2}} \cos x + \frac{1}{\sqrt{2}} \sin x = \frac{1}{\sqrt{2}}$$

∴ $\cos\left(\frac{\pi}{4}\right) \cos x + \sin\left(\frac{\pi}{4}\right) \sin x = \cos\left(\frac{\pi}{4}\right)$

∴ $\cos\left(x - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$ [1]

Since, $\cos\theta = \cos\alpha$ implies $\theta = 2n\pi \pm \alpha$, $n \in Z$. [1]

∴ $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

∴ $x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}$ [1]

∴ $x = 2n\pi + \frac{\pi}{4} + \frac{\pi}{4}$ or $x = 2n\pi - \frac{\pi}{4} + \frac{\pi}{4}$

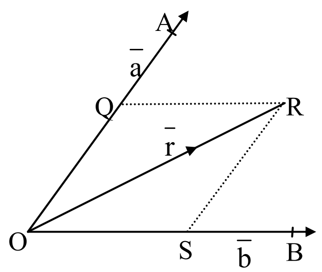
∴ the required general solution is $x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi$, where $n \in Z$. [1]

iii. Suppose $\bar{a}$, $\bar{b}$ and $\bar{r}$ are coplanar.

∴ $\bar{a} = \overline{OA}$, $\bar{b} = \overline{OB}$ and $\bar{r} = \overline{OR}$ are in the same plane, where O is the origin.

Suppose $\bar{r}$ is collinear with either of them, say $\bar{a}$, then

$\bar{r} = t_1\bar{a}$, where t_1 is a scalar.



∴ $\bar{r} = t_1\bar{a} + t_2\bar{b}$ (where $t_2 = 0$) [1]

Now, let $\bar{r}$ be non-collinear with $\bar{a}$ and $\bar{b}$.

Let Q and S lie on the vectors $\bar{a}$ and $\bar{b}$ respectively such that OQRS is a parallelogram.

Now the vectors $\overline{OQ}$ and $\bar{a}$ are collinear and similarly, the vectors $\overline{OS}$ and $\bar{b}$ are collinear.

∴ There exist scalars t_1 and t_2 such that $t_1\bar{a} = \overline{OQ}$ and $t_2\bar{b} = \overline{OS}$.

∴ By parallelogram law of vector addition,

$\overline{OR} = \overline{OQ} + \overline{OS}$.

∴ $\bar{r} = t_1\bar{a} + t_2\bar{b}$. [1]

Thus, $\bar{r}$ can be expressed as a linear combination of $\bar{a}$ and $\bar{b}$.

Conversely, if $\bar{r} = t_1\bar{a} + t_2\bar{b}$ then clearly the vector $\bar{r}$ is in the plane determined by $\bar{a}$ and $\bar{b}$.

∴ $\bar{r}$ is coplanar with $\bar{a}$ and $\bar{b}$.

Uniqueness:

Consider, $\bar{r} = t_1\bar{a} + t_2\bar{b}$ ….(i)

and $\bar{r} = r_1\bar{a} + r_2\bar{b}$ ….(ii)

Subtracting (ii) from (i), we get

$\bar{0} = (t_1 - r_1)\bar{a} + (t_2 - r_2)\bar{b}$

∴ $\bar{a} = \frac{-(t_2 - r_2)}{t_1 - r_1}\bar{b}$ [1]

This shows that the vector $\bar{a}$ is a non-zero scalar multiple of $\bar{b}$.

∴ $\bar{a}$ and $\bar{b}$ are collinear vectors.

This is a contradiction, since $\bar{a}$ and $\bar{b}$ are non-collinear vectors (given)

∴ $t_1 - r_1 = 0$

∴ $t_1 = r_1$

Similarly, $t_2 = r_2$

This shows that $\bar{r}$ is uniquely expressed as a linear combination $t_1\bar{a} + t_2\bar{b}$. [1]

Q.3. (A) Attempt any TWO of the following:

i.

1	2	3	4	5	6
p	q	~q	$p \wedge \sim q$	$p \rightarrow q$	$(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

In the above truth table, all the entries in the last column are F.

∴ $(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$ is a contradiction. [1]

[1 mark each for column 5 and column 6]

ii. The equation of the circle is $S \equiv x^2 + y^2 - 2x + ky - 23 = 0$

Let $P \equiv (x_1, y_1) = (8, -3)$

∴ length of the tangent from (8, –3)

$= \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 - 2x_1 + ky_1 - 23}$ [1]

$= \sqrt{8^2 + (-3)^2 - 2(8) + k(-3) - 23}$

$= \sqrt{64 + 9 - 16 - 3k - 23}$

$= \sqrt{34 - 3k}$

But this is given to be $\sqrt{10}$ units.

∴ $\sqrt{34 - 3k} = \sqrt{10}$ [1]

∴ $34 - 3k = 10$

∴ $3k = 24$

∴ $k = 8$ [1]

iii. Since, direction ratios of lines are not proportional

∴ the given lines are intersecting.

Let $\frac{x+1}{-10} = \frac{y+3}{-1} = \frac{z-4}{1} = \lambda$

∴ the coordinates of a point on this line are $(-10\lambda - 1, -\lambda - 3, \lambda + 4)$

Let $\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4} = \mu$

∴ the coordinates of a point on this line are $(-\mu - 10, -3\mu - 1, 4\mu + 1)$ [1]

Since two lines intersect for some values of λ and μ

∴ $(-10\lambda - 1, -\lambda - 3, \lambda + 4) = (-\mu - 10, -3\mu - 1, 4\mu + 1)$

∴ $-10\lambda - 1 = -\mu - 10$, $-\lambda - 3 = -3\mu - 1$, $\lambda + 4 = 4\mu + 1$

∴ $-10\lambda + \mu = -9$ ….(i)

$-\lambda + 3\mu = 2$ ….(ii)

$\lambda - 4\mu = -3$ ….(iii)

Solving equation (i) and (ii), we get $\lambda = 1$, $\mu = 1$ [1]

and 3rd equation holds for these values.

∴ The lines intersect at the point (–11, –4, 5). [1]

(B) Attempt any TWO of the following:

i. The equation of the hyperbola is $\frac{x^2}{7} - \frac{y^2}{5} = 1$

Here, $a^2 = 7$, $b^2 = 5$

Let $P(x_1, y_1)$ be a point on the locus.

The equation of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with slope m is

$y = mx \pm \sqrt{a^2m^2 - b^2}$

$\therefore \quad y = mx \pm \sqrt{7m^2 - 5}$ [1]

If this tangent passes through $P(x_1, y_1)$, then

$y_1 = mx_1 \pm \sqrt{7m^2 - 5}$

$\therefore \quad y_1 - mx_1 = \pm \sqrt{7m^2 - 5}$

Squaring on both sides, we get

$y_1^2 - 2mx_1y_1 + m^2x_1^2 = 7m^2 - 5$

$\therefore \quad \left(x_1^2 - 7\right)m^2 - 2mx_1y_1 + y_1^2 + 5 = 0$ [1]

$\therefore \quad m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - 7}$ and $m_1m_2 = \frac{y_1^2 + 5}{x_1^2 - 7}$

But, $m_1^3 + m_2^3 = 8$

$\therefore \quad (m_1 + m_2)^3 - 3m_1m_2(m_1 + m_2) = 8$

$\therefore \quad \left(\frac{2x_1y_1}{x_1^2 - 7}\right)^3 - \frac{3\left(y_1^2 + 5\right)\left(2x_1y_1\right)}{\left(x_1^2 - 7\right)^2} = 8$ [1]

$\therefore \quad 8x_1^3y_1^3 - 6x_1y_1\left(y_1^2 + 5\right)\left(x_1^2 - 7\right) - 8\left(x_1^2 - 7\right)^3 = 0$

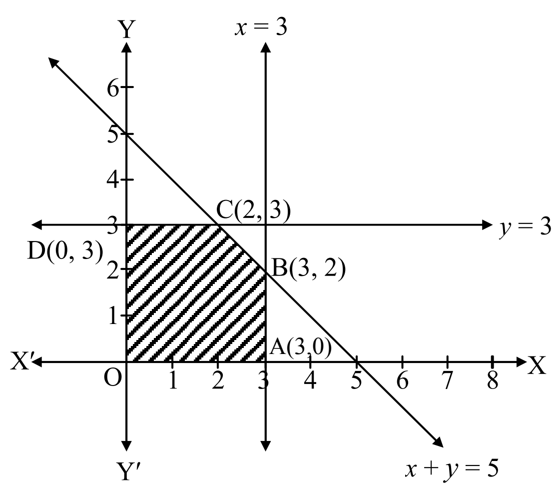
$\therefore$ the equation of the locus of $P(x_1, y_1)$ is

$8x^3y^3 - 6xy(y^2 + 5)(x^2 - 7) - 8(x^2 - 7)^3 = 0$ [1]

ii. To draw the feasible region, construct table as follows:

Inequality	$x \le 3$	$y \le 3$	$x + y \le 5$
Corresponding equation (of line)	$x = 3$	$y = 3$	$x + y = 5$
Intersection of line with X-axis	(3, 0)	–	(5, 0)
Intersection of line with Y-axis	–	(0, 3)	(0, 5)
Region	Origin side	Origin side	Origin side

[1]



[1]

Shaded portion OABCD is the feasible region,
whose vertices are O(0, 0), A(3, 0), B, C and D(0, 3)
B is the point of intersection of the lines $x = 3$ and $x + y = 5$.
Putting $x = 3$ in $x + y = 5$, we get
$y = 2$

∴ B ≡ (3, 2)
C is the point of intersection of the lines $y = 3$ and $x + y = 5$.
Putting $y = 3$ in $x + y = 5$, we get
$x = 2$

∴ C ≡ (2, 3) [1]

Here, the objective function is
$Z = 10x + 25y$

∴ Z at O(0, 0) = 10(0) + 25(0) = 0
Z at A(3, 0) = 10(3) + 25(0) = 30
Z at B(3, 2) = 10(3) + 25(2) = 30 + 50 = 80
Z at C(2, 3) = 10(2) + 25(3) = 20 + 75 = 95
Z at D(0, 3) = 10(0) + 25(3) = 75

∴ Z has maximum value 95 at C(2, 3).

∴ Z is maximum, when $x = 2$ and $y = 3$. [1]

iii. The equation of the plane parallel to the plane $x + 2y + 2z + 8 = 0$ is
$x + 2y + 2z + \lambda = 0$ ….(i) [1]

Distance of the plane (i) from the point (1, 1, 2) is given by

$$\left|\frac{1+2(1)+2(2)+\lambda}{\sqrt{1^2+2^2+2^2}}\right| = \left|\frac{1+2+4+\lambda}{\sqrt{1+4+4}}\right| = \left|\frac{7+\lambda}{3}\right|$$ [1]

But this distance is given to be 2.

∴ $\left|\frac{\lambda+7}{3}\right| = 2$

∴ $\frac{\lambda+7}{3} = \pm 2$

∴ $\lambda = -13$ or $\lambda = -1$ [1]

Putting the value of λ in (i), we get
$x + 2y + 2z - 13 = 0$ or $x + 2y + 2z - 1 = 0$ which are the equations of planes. [1]

Section – II

Q.4. (A) Select and write the correct answer from the given alternatives in each of the following:

i. (D)
$f'(x) = 2x - 3$
$f(x)$ has minimum value at $x = c$, if
$f'(c) = 0$ and $f''(c) > 0$

∴ $2x - 3 = 0$ and $f''(x) = 2 > 0$

∴ $x = \frac{3}{2}$ [2]

ii. (B)

Put $\log x = t$

$\therefore \quad \frac{1}{x} dx = dt$

$$\therefore \quad \int \frac{1}{x} . \log x \, dx = \int t dt = \frac{t^2}{2} + c = \frac{1}{2}(\log x)^2 + c$$ [2]

iii. (A)

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{7}{3}} = 7 \cdot \frac{d^2 y}{dx^2}$$

Cubing on both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{7} = 7^3\left(\frac{d^2 y}{dx^2}\right)^3$$

By definition of degree and order

Degree: 3 ; Order: 2 [2]

(B) Attempt any THREE of the following:

i. $x = at^2$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}\left(at^2\right) = a\frac{d}{dt}\left(t^2\right) = 2at$$

$y = 2at$

Differentiating w.r.t. t, we get

$$\frac{dy}{dt} = \frac{d}{dt}\left(2at\right) = a\frac{d}{dt}\left(2t\right) = 2a$$ [1]

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2a}{2at} = \frac{1}{t}$$ [1]

ii. Let $f(x) = \sqrt{x}$

$\therefore \quad f'(x) = \frac{1}{2\sqrt{x}}$

$x = 8.95 = 9 - 0.05 = a + h$

Here, $a = 9$ and $h = -0.05$

$f(a) = f(9) = \sqrt{9} = 3$ and

$f'(a) = f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$ [1]

$f(a + h) \approx f(a) + hf'(a)$

$\therefore \quad \sqrt{8.95} \approx 3 + (-0.05)\left(\frac{1}{6}\right) \approx 3 - 0.0083$

$\therefore \quad \sqrt{8.95} \approx 2.9917$ [1]

iii. Given equation of parabola is $y^2 = 16x$.

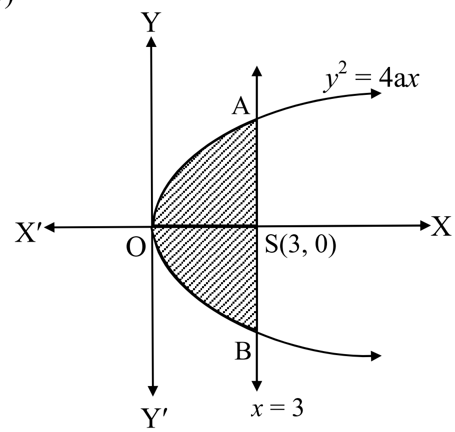
$\therefore$ $y = \pm 4\sqrt{x}$

$\therefore$ required area = area of the region OBSAO

= 2(area of the region OSAO)

$= 2\int_0^3 y\, dx$

$= 2\int_0^3 4\sqrt{x}\, dx$ [1]



$= 8\left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$

$= \dfrac{16}{3}\left(3^{\frac{3}{2}} - 0\right)$

$= 16\sqrt{3}$ sq. units [1]

iv. Given, r = 0.3, cov (X, Y) = 18, $\sigma_x = 3$

Now, $r = \dfrac{\text{cov}(X, Y)}{\sigma_x . \sigma_y}$

$\therefore$ $0.3 = \dfrac{18}{3(\sigma_y)}$ [1]

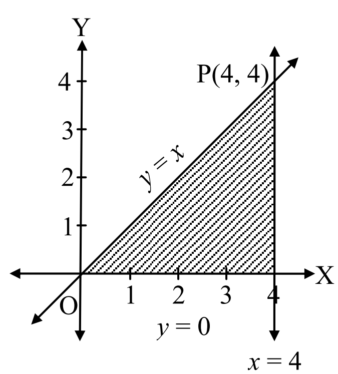
$\therefore$ $\sigma_y = \dfrac{18}{0.9} = 20$ [1]

v. Given lines are $y = x$, $x = 4$ and $y = 0$.

Let the volume of solid of revolution be V, which is revolved about the X-axis, between $x = 0$ to $x = 4$.

$\therefore$ $V = \pi \int_0^4 y^2\, dx$

$= \pi \int_0^4 x^2\, dx$ [1]



$= \pi \left[\dfrac{x^3}{3}\right]_0^4$

$= \dfrac{\pi}{3}(4^3 - 0)$

$= \dfrac{\pi}{3}(64)$

$= \dfrac{64\pi}{3}$ cubic units. [1]

Q.5. (A) Attempt any TWO of the following:

i. f(x) is continuous in its domain

$\therefore$ f(x) is continuous at $x = 0$

$\therefore$ $\lim\limits_{x\to0^-} f(x) = f(0) = \lim\limits_{x\to0^+} f(x)$

$\therefore$ $\lim\limits_{x\to0^-} (x + a) = \lim\limits_{x\to0^+} x$

$\therefore$ $0 + a = 0$

$\therefore$ $a = 0$ [1]

Also, f(x) is continuous at $x = 1$.

$\therefore$ $\lim\limits_{x\to1^-} f(x) = f(1) = \lim\limits_{x\to1^+} f(x)$

$\therefore$ $\lim\limits_{x\to1^-} x = \lim\limits_{x\to1^+} (b - x)$

$\therefore$ $1 = b - 1$

$\therefore$ $b = 2$ [1]

$\therefore$ $a + b = 0 + 2 = 2$ [1]

ii. $x = a\left(t - \frac{1}{t}\right)$

$\therefore$ $\frac{x}{a} = t - \frac{1}{t}$

Squaring on both sides, we get

$\frac{x^2}{a^2} = t^2 - 2 + \frac{1}{t^2}$

$y = a\left(t + \frac{1}{t}\right)$

$\therefore$ $\frac{y}{a} = t + \frac{1}{t}$

Squaring on both sides, we get

$\frac{y^2}{a^2} = t^2 + 2 + \frac{1}{t^2}$ [1]

$\therefore$ $\frac{x^2}{a^2} - \frac{y^2}{a^2} = t^2 - 2 + \frac{1}{t^2} - t^2 - 2 - \frac{1}{t^2} = -4$

$\therefore$ $x^2 - y^2 = -4a^2$ [1]

Differentiating w.r.t. x, we get

$2x - 2y\frac{dy}{dx} = 0$

$\therefore$ $2y\frac{dy}{dx} = 2x$

$\therefore$ $\frac{dy}{dx} = \frac{x}{y}$ [1]

iii. Let $I = \int \frac{1}{3+5\cos x}\,dx$

Put $\tan\left(\frac{x}{2}\right) = t$

$\therefore$ $x = 2\tan^{-1} t$

$\therefore$ $dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$ [1]

$\therefore$ $I = \int \frac{1}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$

$= \int \frac{2}{3+3t^2+5-5t^2}\,dt$

$= 2\int \frac{dt}{-2t^2+8} = \int \frac{dt}{-t^2+4}$

$= \int \frac{dt}{2^2-t^2}$ [1]

$= \frac{1}{2\times 2}\log\left|\frac{2+t}{2-t}\right| + c$

$\therefore$ $I = \frac{1}{4}\log\left|\frac{2+\tan\left(\frac{x}{2}\right)}{2-\tan\left(\frac{x}{2}\right)}\right| + c$ [1]

(B) Attempt any TWO of the following:

i. Let X be the number of men who survive the next 30 years.

P(the man of this age will survive the next 30 years) = $p = \frac{2}{3}$

$\therefore$ $q = 1 - p = 1 - \frac{2}{3} = \frac{1}{3}$ [1]

Given n = 5

$\therefore$ $X \sim B\left(5, \frac{2}{3}\right)$

The p.m.f. of X is given by

$P(X = x) = p(x) = {}^5C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{5-x}$, $x = 0,1,2,3,4,5$ [1]

P(at most three men will survive in the next 30 years)

$= P(X \le 3)$

$= 1 - P(X > 3)$

$= 1 - [P(X = 4) + P(X = 5)]$ [1]

$= 1 - \left[{}^5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + {}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0\right]$

$= 1 - \left(\frac{80}{243} + \frac{32}{243}\right)$

$= 1 - \frac{112}{243}$

$= 1 - 0.4609$

$\therefore$ P(at most three men will survive in the next 30 years) = 0.5391 [1]

ii. Let r be the radius of the spherical balloon.

∴ $r = 6$ cm ….(given)

Let s be the surface area of the spherical ballon

∴ $s = 4\pi r^2$ …. (i)

Surface area of a spherical balloon is increasing at the rate of 2 cm^2/sec.

i.e., $\frac{ds}{dt} = 2\ cm^2/sec$ ….(given) [1]

Differentiating (i) w.r.t. t, we get

$\frac{ds}{dt} = 4\pi(2r)\cdot\frac{dr}{dt}$

∴ $2 = 8\pi r \cdot \frac{dr}{dt}$

∴ $2 = 8\pi(6) \cdot \frac{dr}{dt}$

∴ $\frac{1}{24\pi} = \frac{dr}{dt}$ ….(ii) [1]

Let V be the volume of spherical balloon.

Then, $V = \frac{4}{3}\pi r^3$

∴ $\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\cdot\frac{dr}{dt}$

$= 4\pi r^2\left(\frac{dr}{dt}\right)$ [1]

∴ $\frac{dV}{dt} = 4\pi(6)^2\cdot\frac{1}{24\pi}$ ….[From (ii)]

$= 4\pi\cdot 36 \times \frac{1}{24\pi}$

∴ $\frac{dV}{dt} = 6\ cm^3/sec$

Thus, the volume of the spherical balloon is increasing at the rate of 6 cm^3/sec. [1]

iii. Since, $\frac{dy}{dx}$ represents the slope of tangent to a given curve at a point (x, y), the given equation is $\frac{dy}{dx} = y + 2x$

∴ $\frac{dy}{dx} - y = 2x$ [1]

The given equation is of the form $\frac{dy}{dx} + Py = Q$,

where $P = -1$ and $Q = 2x$

∴ $I.F. = e^{\int P dx} = e^{\int -1.dx} = e^{-x}$ [1]

∴ Solution of the given equation is

$y\,(I.F.) = \int Q\,(I.F.)\,dx + c$

$\therefore$ $ye^{-x} = \int 2x\, e^{-x}\, dx + c$

$= 2x\int e^{-x} dx - \int\left[\frac{d}{dx}(2x)\int e^{-x}dx\right]dx + c = -2x\,e^{-x} + 2\int e^{-x}\,dx + c$

$= -2xe^{-x} - 2e^{-x} + c$

$\therefore$ $ye^{-x} = -2(x+1)\,e^{-x} + c$

$\therefore$ $y = -2(x+1) + ce^{x}$ …..(i) [1]

Since, the required curve passes through origin

$\therefore$ $x = 0$ and $y = 0$

$\therefore$ Equation (i) becomes

$0 = -2(0+1) + ce^{0}$

$\therefore$ $c = 2$

$\therefore$ Required equation of curve is

$y = -2(x+1) + 2e^{x}$

$\therefore$ $y + 2(x+1) = 2e^{x}$ [1]

Q.6. (A) Attempt any TWO of the following:

i. Let $\int v\,dx = w$

$\therefore$ $\frac{dw}{dx} = v$

Consider, $\frac{d}{dx}(u\cdot w) = u\frac{dw}{dx} + w\frac{du}{dx}$ [1]

$= uv + \frac{du}{dx}\int v\,dx$

Integrating on both sides w.r.t. 'x', we get

$u\cdot w = \int u\,v dx + \int\left(\frac{du}{dx}\int v\,dx\right)dx$ [1]

$\therefore$ $u\int v\ dx = \int uv\,dx + \int\left(\frac{du}{dx}\int v\,dx\right)dx$

$\therefore$ $\int u\cdot v\,dx = u\int v\,dx - \int\left(\frac{d}{dx}(u)\cdot\int v\,dx\right)dx$ [1]

ii. Let X = Preparation time (in minutes)

$P(X > 33) = \int_{33}^{\infty} f(x)\,dx$

$= \int_{33}^{35} f(x)\,dx + \int_{35}^{\infty} f(x)\,dx$

$= \int_{33}^{35} f(x)\,dx + 0$ ….[$\because$ $f(x) = 0$, when $x > 35$]

$= \int_{33}^{35} \frac{1}{10}\,dx$

$= \frac{1}{10}[x]_{33}^{35} = \frac{35-33}{10}$

$= \frac{1}{5}$ [1]

Let $F(x)$ be the c.d.f. of X.

$\therefore$ $F(x) = P(X \le x)$

$$= \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{25} f(x)dx + \int_{25}^{x} f(x)dx$$ [1]

$$= 0 + \int_{25}^{x} f(x)\,dx \quad \text{....}[\because f(x) = 0, \text{ when } x < 25]$$

$$= \int_{25}^{x} \frac{1}{10} dx$$

$$= \frac{1}{10}[x]_{25}^{x}$$

$\therefore$ $F(x) = \dfrac{x-25}{10}$ [1]

iii. Let X be the number of tested components survive.

P(component survive the check test) = p = 0.6

$\therefore$ q = 1 – 0.6 = 0.4

Given n = 4

$\therefore$ X ~ B (4, 0.6)

The p.m.f. of X is given by

$P(X = x) = p(x) = {}^4C_x (0.6)^x \ (0.4)^{4-x}$, $x = 0,1,\ldots, 6$ [1]

$\therefore$ P(exactly 2 components survive) = P(X = 2) $= {}^4C_2 (0.6)^2 (0.4)^{4-2}$ [1]

$= {}^4C_2 (0.6)^2 (0.4)^2$

$= 6(0.36)(0.16)$

$= 0.3456$ [1]

(B) Attempt any TWO of the following:

i. $ax^2 + 2hxy + by^2 = 0$ ....(i)

Differentiating w.r.t.x, we get

$$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$$

$\therefore$ $ax + hx\dfrac{dy}{dx} + by\dfrac{dy}{dx} + hy = 0$ [1]

$\therefore$ $(hx + by)\dfrac{dy}{dx} = -(ax + hy)$

$\therefore$ $\dfrac{dy}{dx} = -\dfrac{ax+hy}{hx+by}$ ....(ii) [1]

From (i), we have $ax^2 + hxy + hxy + by^2 = 0$

$\therefore$ $x(ax + hy) + y(hx + by) = 0$

$\therefore$ $-\dfrac{ax+hy}{hx+by} = \dfrac{y}{x}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{y}{x}$ ....[From (ii)] [1]

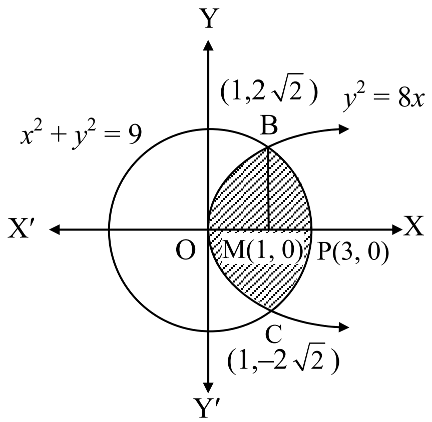
Again, differentiating w.r.t.x, we get

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y}{x^2}$$

$$= \frac{x\left(\frac{y}{x}\right) - y}{x^2} = \frac{y - y}{x^2}$$

$$= 0$$ [1]

ii.



Equation of circle is $x^2 + y^2 = 9$ ....(i)
and equation of parabola is $y^2 = 8x$ ....(ii)
To find the points of intersection of (i) and (ii).
Substituting (ii) in (i), we get
$x^2 + 8x - 9 = 0$

$\therefore$ $(x + 9)(x - 1) = 0$

$\therefore$ $x = -9$ or $x = 1$

When $x = -9$, $y^2 = -72$, which is not possible
When $x = 1$, $y = \pm\ 2\sqrt{2}$

$\therefore$ the points of intersection of the two given curves are B $\left(1, 2\sqrt{2}\right)$ and C$\left(1, -2\sqrt{2}\right)$. [1]

$\therefore$ required area = area of the region OCPBO
= 2(area of the region OMPBO)
= 2(area of the region OMBO + area of the region BMPB) [1]

$$= 2\left[\int_0^1 2\sqrt{2x}\,dx + \int_1^3 \sqrt{9-x^2}\,dx\right] \quad \text{....[From (i) and (ii)]}$$

$$= 4\sqrt{2}.\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 + 2.\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_1^3$$ [1]

$$= \frac{8\sqrt{2}}{3}(1 - 0) + 2\left[\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(1) - \frac{1}{2}\sqrt{8} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{8\sqrt{2}}{3} + 9.\left(\frac{\pi}{2}\right) - 2\sqrt{2} - 9\sin^{-1}\left(\frac{1}{3}\right)$$

$\therefore$ required area = $\left[\frac{8\sqrt{2}}{3} + \frac{9\pi}{2} - 2\sqrt{2} - 9\sin^{-1}\left(\frac{1}{3}\right)\right]$ sq. units. [1]

iii. Let $X = x_i$ and $Y = y_i$, $u_i = x_i - 2$, $v_i = y_i - 5$

$n = 10$, $\sum u_i = 30$, $\sum v_i = 40$, $\sum u_i^2 = 900$, $\sum v_i^2 = 800$, $\sum u_i v_i = 480$ ….(given)

$\therefore \quad \bar{u} = \frac{\sum u_i}{n} = \frac{30}{10} = 3, \ \bar{v} = \frac{\sum v_i}{n} = \frac{40}{10} = 4$ [1]

$$\text{Corr}(U, V) = \frac{\frac{1}{n}\Sigma u_i v_i - \bar{u}\cdot\bar{v}}{\sqrt{\frac{1}{n}\Sigma u_i^2 - \left(\bar{u}\right)^2}\cdot\sqrt{\frac{1}{n}\Sigma v_i^2 - \left(\bar{v}\right)^2}}$$ [1]

$$= \frac{\frac{480}{10} - (3)(4)}{\sqrt{\frac{900}{10} - (3)^2}\cdot\sqrt{\frac{800}{10} - (4)^2}}$$ [1]

$$= \frac{48 - 12}{\sqrt{81}\cdot\sqrt{64}} = \frac{36}{9\times 8} = 0.5$$

$\therefore$ Corr (X, Y) = Corr (U, V) = 0.5 [1]