7192/M11 **OCTOBER 2008**

Paper I — CALCULUS AND CLASSICAL ALGEBRA

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

SECTION A — (8 × 5 = 40 marks)

Answer any EIGHT questions.

1. If $y = \cos x \cos 2x \cos 3x$, find y_n.

2. Expand $\log_e (1+x)$ as an infinite series.

3. Find the equation of the tangent at the point (3, 1) on the curve $x^2 + 5y^2 = 14$.

4. Find the envelope of the family of circles $(x-a)^2 + y^2 = 2a$, where a is the parameter.

5. Prove that $\displaystyle\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\,dx = \frac{\pi}{4}$.

6. Find the reduction formula for $I_n = \int x^n \cos ax\,dx$ (x is a positive integer).

7. Prove that a sequence cannot converge to two different limits.

8. Test the convergence of the series

$$\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\cdots.$$

9. Show that any absolutely convergent series is convergent.

10. Sum the series,

$$1+\frac{1+3}{2!}+\frac{1+3+3^2}{3!}+\frac{1+3+3^2+3^3}{4!}+\cdots.$$

11. Evaluate : $\lim_{x\to 0}\frac{e^x-e^{-x}}{\log(1+x)}$.

12. Find the sum to x terms of the series $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots$

SECTION B — (6 × 10 = 60 marks)

Answer any SIX questions.

13. If $y=a\cos(\log x)+b\sin(\log x)$, prove that $x^2y_{n+2}+(2n+1)x\,y_{n+1}+(n^2+1)y_n=0$.

14. Find the points in the curve $y=x^4-6x^3+13x^2-10x+5$, where the tangents are parallel to $y=2x$ and prove that two of these points have the same tangent.

15. Show that the evolute of the parabola $y^2=4ax$ is $27ay^2=4(x-2a)^3$.

16. Find the reduction formula for $\int \sin^n x\,dx$. and hence the value of $\int_0^{\pi/2}\sin^n x\,dx$, where x is a positive integer.

17. Find the Fourier series expansion of $f(x)=\frac{1}{2}(\pi-x)$ in the interval $(0,2\pi)$.

18. State and prove comparison test.

19. State and prove Cauchy's integral test.

20. Sum the series

$$\frac{5}{3\cdot 6}+\frac{5\cdot 7}{3\cdot 6\cdot 9}+\frac{5\cdot 7\cdot 9}{3\cdot 6\cdot 9\cdot 12}+\cdots$$

21. Show that, if $x > 0$

$$\log x = \frac{x-1}{x+1} + \frac{1}{2} \cdot \frac{x^2-1}{(x+1)^2} + \frac{1}{3} \cdot \frac{x^3-1}{(x+1)^3} + \cdots$$

22. Sum the series $\sum_{n=0}^{\infty} \frac{(n+1)^3}{x!} x^n$.

18. Find the centre and the radius of the circle of intersection of the plane $2x - 2y + z = 27$ and $x^2 + y^2 + z^2 = 225$.

19. Find the volume of the tetrahedron formed by the planes whose equations are $y + z = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = 1$.

20. With usual notations, prove the following :

(a) $curl\left(\vec{F} + \vec{G}\right) = curl\ \vec{F} + curl\ \vec{G}$.

(b) $Div\left(\vec{F} \times \vec{G}\right) = \vec{G}.curl\ \vec{F} - \vec{F}.curl\ \vec{G}$.

21. Evaluate $\iint_S \vec{F}.\vec{n}\, ds$ where $\vec{F} = 4xz\ \vec{i} - y^2\ \vec{j} + yz\ \vec{k}$ and S is the surface of the cube bounded by $x = 0, x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

22. Verify Stoke's theorem for the function $\vec{F} = x^2\ \vec{i} + xy\ \vec{j}$ integrated around the sides $x = 0,\ y = 0,\ x = a,\ y = a$ of a square in the $z = 0$ plane.

Paper II — TRIGONOMETRY, ANALYTICAL GEOMETRY OF THREE DIMENSIONS AND VECTOR CALCULUS

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

SECTION A — (8 × 5 = 40 marks)

Answer any EIGHT questions

1. Prove that the three points representing $5 + 8i$ $13 + 20i$ and $19 + 29i$ are collinear.

2. If $\tan(x + iy) = u + iv$ prove that $\dfrac{u}{v} = \dfrac{\sin 2x}{\sinh 2y}$.

3. Find the equation of the plane passing through $(2,2,1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$.

4. Find the point where the line $\dfrac{x-2}{2} = \dfrac{y-4}{-3} = \dfrac{z+6}{-4}$ meets the plane $2x + 4y - z - 2 = 0$.

5. Find the angle between the line $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-3}{-2}$ and the plane $x + 2y + z - 3 = 0$.

6. Find the equation to the plane containing the line $\frac{x-1}{1} = \frac{y}{2} = \frac{z+1}{3}$ and parallel to the line $\frac{x-2}{3} = \frac{y-4}{-1} = \frac{z-5}{2}$.

7. Find the equation of the sphere which has its centre at $(6, -1, 2)$ and touches the plane $2x - y + 2z - 2 = 0$.

8. Find the equation of the cone whose vertex is $(1, 2, 3)$ and guiding curve is the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 1$.

9. Prove that the directional derivative of $f(x, y, z) = z^2 x + y^3$ at $(1, 1, 2)$ in the direction $\left[\frac{1}{\sqrt{5}}\right]\vec{i} + \left[\frac{2}{\sqrt{5}}\right]\vec{j}$ is $2\sqrt{5}$.

10. Prove that $\frac{y}{x^2 + y^2}\vec{i} - \frac{x}{x^2 + y^2}\vec{j}$ is irrotational.

11. Compute $\int_D \sin(x + y)\, dx\, dy$ where $D = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$.

12. If S is any closed surface enclosing a volume V and $\vec{F} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$, prove that $\int_S \vec{F} \cdot \vec{dS} = (a + b + c)V$.

SECTION B — (6 × 10 = 60 marks)

Answer any SIX questions.

13. Express $\cos^4 \theta \sin^3 \theta$ in a series of sines of multiples of θ.

14. Find the sum to n terms of the series $\sin \alpha + c \sin(\alpha + \beta) + c^2 \sin(\alpha + 2\beta) + \ldots$.

15. A plane meets the coordinate axis at A, B, C such that the centroid of the ΔABC is the point (α, β, λ). Prove that the equation of the plane is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\lambda} = 3$.

16. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in A, B, C. Find the coordinates of the orthocentre of the triangle ABC.

17. Find the shortest distance between the lines

$$\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}; \quad \frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}.$$