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Entrance Examination : M.Sc. Mathematics, 2013

Hall Ticket Number [| | | | | | |]

Time : 2 hours
Max. Marks. 100

Part A : 25 marks
Part B : 75 marks

Instructions

1. Write your Hall Ticket Number on the OMR Answer Sheet given to you. Also write the Hall Ticket Number in the space provided above.

2. In part A a right answer gets 1 **mark** and a wrong answer gets **−0.33 mark.**

3. **In Part B, some questions have MORE THAN ONE correct option. All the correct options have to be marked in the OMR answer sheet, otherwise ZERO marks will be credited.**

4. Answers are to be marked on the OMR answer sheet following the instructions provided there upon.

5. Hand over the OMR answer sheet at the end of the examination.

6. No additional sheets will be provided. Rough work can be done in the question paper itself/space provided at the end of the booklet.

7. There are a total of 50 questions in Part A and Part B together.

8. The appropriate answer should be coloured in either a blue or black ball point or sketch pen. DO NOT USE A PENCIL.

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Part A

1. We say that a sequence (a_n) does NOT converge to l if

   A. $\forall \varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \forall n \geq n_0$ we have $|a_n - l| > \varepsilon$.
   B. $\forall \varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \geq n_0$ such that $|a_n - l| > \varepsilon$.
   C. $\exists \varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n > n_0$ such that $|a_n - l| > \varepsilon$.
   D. $\exists \varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \forall n \geq n_0$ we have $|a_n - l| > \varepsilon$.

2. Consider a sequence (a_n) of positive numbers satisfying the condition $a_n a_{n+2} \leq a_{n+1}^2,\ \forall n \in \mathbb{N}$ then (a_n) is a

   A. convergent sequence if $a_1 \neq 2a_2$.
   B. monotonically increasing sequence if $a_1 \neq 2a_2$.
   C. convergent sequence if $a_1 = 2a_2$.
   D. monotonically increasing sequence if $a_1 = 2a_2$.

3. The sum of the series $\sum_{n=1}^{\infty} \left[(n+1)^{\frac{1}{5}} - n^{\frac{1}{5}}\right]$ is

   A. less than -1.
   B. equal to -1.
   C. greater than -1 but less than 2.
   D. none of the above.

4. Let $S = \{x \in \mathbb{R}/x^2 \leq 5\} \cap \mathbb{Q}$. Which of the following statements is true about S?

   A. S is bounded above and $\sup S \in \mathbb{Q}$.
   B. S is bounded above and $\sup S \in \mathbb{R} - \mathbb{Q}$.
   C. S is a closed interval.
   D. S is an open interval.

5. The value of $\lim_{x \to 0} \dfrac{e^{(1/x)} - e^{(-1/x)}}{e^{(1/x)} + e^{(-1/x)}}$ is

   A. 0.
   B. 1.
   C. -1.
   D. none of the above.

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} -x+3, & x \in \mathbb{Q}, \\ x^2 - 6x + 9 & x \notin \mathbb{Q}. \end{cases}$

   The set of all points at which f is continuous is

   A. $\{2, 3\}$. **B.** $\{3\}$. **C.** $\mathbb{R} - \{2, 3\}$. **D.** $\mathbb{R} - \{3\}$.

7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} \sin x, & x \geq 0, \\ 1 - \cos x, & x < 0. \end{cases}$
Which of the following statements is true about f?
 A. f is differentiable.
 B. f is continuous but NOT differentiable.
 C. f is discontinuous.
 D. none of the above statements is true.

8. Let $f : [0,1] \to \mathbb{R}$, $g : [0,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0 & x \notin \mathbb{Q} \cap [0,1], \end{cases}$
and $g(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap [0,1], \\ 1 & x \notin \mathbb{Q} \cap [0,1], \end{cases}$ then
 A. both f and g are Riemann integrable.
 B. f is Riemann integrable but g is NOT Riemann integrable.
 C. g is Riemann integrable but f is NOT Riemann integrable.
 D. both f and g are NOT Riemann integrable.

9. $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{2k}{k^2 + n^2} =$
 A. 0 B. $\log 2$ C. 2 D. ∞

10. A solution of $xdy - ydx + (x^2 + y^2)dx + (x^2 + y^2)dy = 0$ is
 A. $\arctan(y/x) + x + y = C$. B. $\frac{y}{x} + x^2 + y^2 = C$.
 C. $\arctan(y/x) + x^2 + y^2 = C$. D. $\frac{y}{x} + x + y = C$.

11. The general solution of $(D^4 + I)^2 y = 0$ is
 A. $C_1 \sin x + C_2 \cos x + C_3 e^x + C_4 e^{-x}$.
 B. $C_1 x \sin x + C_2 x \cos x + C_3 e^x + C_4 e^{-x}$.
 C. $(C_1 + C_2 x)\sin x + (C_3 + C_4 x)\cos x + C_5 e^x + C_6 e^{-x}$.
 D. $(C_1 + C_2 x)\sin x + (C_3 + C_4 x)\cos x + (C_5 + C_6 x)e^x + (C_7 + C_8 x)e^{-x}$.

12. Consider three different planes $a_{11}x + a_{12}y + a_{13}z = d_1$, $a_{21}x + a_{22}y + a_{23}z = d_2$ and $a_{31}x + a_{32}y + a_{33}z = d_3$. Let $A = (a_{ij})$, $1 \leq i, j \leq 3$. Which of the following conditions necessarily implies that there exists a unique point of intersection of all three planes?
 A. $\det(A) = 0$ B. $\det(A) \neq 0$
 C. $\mathrm{Trace}(A) = 0$ D. $\mathrm{Trace}(A) \neq 0$

13. The number of planes containing both the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x-2}{-1} = \frac{y-4}{-5} = \frac{z-6}{-1}$ is

A. 0.
B. 1.
C. more than 1 but finitely many.
D. infinite.

14. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 22 \\ 0 & 1/7 & \pi \end{bmatrix}$, then $\det(A)$ is

A. zero.
B. a nonzero rational number.
C. an irrational number less than 1.
D. an irrational number greater than 1.

15. Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$ and $A, B \subset \mathbb{R}^3$ such that $0 \notin A \cup B$. Let the number of elements in A and B are 4 and 2 respectively, then

A. both A and B are linearly dependent sets.
B. A is linearly dependent set but B is linearly independent set.
C. both A and B are linearly independent sets.
D. none of the above is a true statement.

16. The number of group homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{13}$ is

A. 0.
B. 1.
C. more than 1 but finitely many.
D. infinite.

17. The center of $\mathbb{Z}_{33}$ is

A. $\{0\}$.
B. $\mathbb{Z}_3$.
C. $\mathbb{Z}_{11}$.
D. $\mathbb{Z}_{33}$.

18. Let G be a group and H be a subgroup of G. Which of the following statements is true?

A. If H is a normal subgroup of G then $gH = Hg, \forall g \in G$.
B. If H is a normal subgroup of G then $gH \neq Hg$, for some g $\in$ G.
C. If $gH = Hg$, for some $g \in G$ then H is a normal subgroup of G.
D. If $gH \neq Hg$, for some g $\in$ G then H is a normal subgroup of G.

19. The number of elements of order 8 in a cyclic group of order 16 is

A. 1.
B. 2.
C. 3.
D. 4.

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20. If $x \neq e$, $y \neq e$ are elements in a group G such that the order of x is 2 and $x^{-1}yx = y^2$ then the order of y is

A. 1. B. 2. C. 3. D. 4.

21. In the ring $(\mathbb{Z}, +, .)$ the set $\{12u + 30v | u, v \in \mathbb{Z}\}$ is same as $n\mathbb{Z}$ for $n =$

A. 6. B. 4. C. 3. D. 2.

22. Let S be the sphere with center at the origin and radius 1. Let $\bar{f}$ is a vector field given by $\bar{f}(x, y, z) = (z - 2xyz)\hat{i} + 9x^2yz^2\hat{j} + (yz^2 - 3x^2z^3)\hat{k}$. If $\hat{n}$ is the outward normal then, the value of $\iint_S \bar{f}.\hat{n}dS =$

A. 0. B. $\frac{4}{3}\pi$. C. π. D. $\frac{4}{3}\pi^3$.

23. If ϕ is a real valued smooth function and $\bar{f}$ is a vector valued smooth function on $\mathbb{R}^3$, then $\text{div}(\phi \ \text{Curl}\bar{f}) =$

A. $\nabla\phi.\ \text{Curl}\bar{f}$
B. $\nabla(\bar{f}.\nabla\phi)$
C. $\nabla\phi.\ \text{Curl}\bar{f} + \nabla(\bar{f}.\nabla\phi)$
D. none of the above.

24. What is the probability of that girls outnumber boys in a family with 5 children. Assume that births are independent trials and probability of a boy is equal to 1/2.

A. 0. B. $\frac{1}{2}$. C. $\frac{15}{32}$. D. $\frac{17}{32}$.

25. Consider two boxes numbered Box1 and Box2. Let Box1 contains 5 red balls and 4 black balls Box2 contains 10 red balls and 17 black balls. Consider a random experiment of choosing a box, picking a ball from it. What is the probability that the color of the ball is red?

A. $\frac{25}{54}$ B. $\frac{50}{54}$ C. $\frac{15}{36}$ D. $\frac{15}{17}$.

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Part - B

Correct answer/s marked in OMR sheet to a question in this Section get 3 marks and zero otherwise.

26. Consider the statement 'There is a train in which every compartment has at least one passenger without the ticket.' Negation of this statement is

    A. There is a train in which every compartment has at least one passenger with the ticket.
    B. There is a train in which every passenger of every compartment has the ticket.
    C. Every train has a compartment in which every passenger has the ticket.
    D. In every train every passenger in every compartment has the ticket.

27. Consider a sequence (a_n) of real numbers. Which of the following conditions imply that (a_n) is convergent?

    A. $|a_{n+1} - a_n| < \frac{1}{n}, \forall n \in \mathbb{N}$.

    B. $|a_{n+1} - a_n| < \frac{1}{3^n}, \forall n \in \mathbb{N}$.

    C. $a_n > 0, \forall n \in \mathbb{N}$ and a_n is monotonically increasing.

    D. $a_n > 0, \forall n \in \mathbb{N}$ and a_n is monotonically decreasing.

28. Which of the following series are convergent?

    A. $\sum_{n=0}^{\infty} \frac{\log n}{n^{3/2}}$ B. $\sum_{n=0}^{\infty} \frac{n^2}{n!}$. C. $\sum_{n=0}^{\infty} \frac{1}{n \log n}$ D. $\sum_{n=0}^{\infty} \frac{e^n}{n^{100}}$.

29. Which of the following statements are true?

    A. If $A \subset \mathbb{Q}$ such that $\mathbb{Q} - A$ is finite then A is dense in $\mathbb{R}$.
    B. There exists $A \subset \mathbb{Q}$ such that $\mathbb{Q} - A$ is infinite and A is dense in $\mathbb{R}$.
    C. There exists a pair of disjoint subsets of Q such that both of them are dense in $\mathbb{R}$.
    D. None of the above is a true statement.

30. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin^3(|x|)$, then $f'(0)$

    A. is equal to -1. B. is equal to 0.
    C. is equal to 1. D. does not exist.

31. Consider the following two statements.

S_1: If $f:[0,1]\to[0,1]$ is continuous then $\exists x_0\in[0,1]$ such that $f(x_0)=x_0$.

S_2: There exists a continuous function $f:[0,1]\to[0,1]-\{\frac{1}{2}\}$ such that f is on to.

A. Both S_1 and S_2 are true.
B. S_1 is true but S_2 is FALSE.
C. S_2 is true but S_1 is FALSE.
D. Both S_1 and S_2 are FALSE.

32. Consider the following two statements.

S_1: $\int_0^{\pi/2}\frac{\sin x}{x}dx$ exists.

S_2: $\int_0^1 \frac{x}{\log x}dx$ exists.

A. Both S_1 and S_2 are true.
B. S_1 is true but S_2 is FALSE.
C. S_2 is true but S_1 is FALSE.
D. Both S_1 and S_2 are FALSE.

33. Solution of $(x^2+y^2)xdx+(x^2+y^2)ydy+2xy(xdy-ydx)=0$ is

A. $\log(\sqrt{x^2+y^2})-\frac{x^2}{x^2+y^2}=C.$ B. $\log(x^2+y^2)-\frac{x^2}{x^2+y^2}=C.$

C. $\log(\sqrt{x^2+y^2})-\tan^{-1}\frac{y}{x}=C.$ D. $\log(x^2+y^2)-\tan^{-1}\frac{y}{x}=C.$

34. The general solution of $(D^2-I)y=x^2+e^{-x}$ is

A. $C_1e^x+C_2e^{-x}-\left[\frac{1}{4}(2x+1)e^{-x}+x^2+2\right].$
B. $C_1\sin x+C_2\cos x-\left[\frac{1}{4}(2x+1)e^{-x}+x^2+2\right].$
C. $C_1e^x+C_2e^{-x}-\left[\frac{1}{2}e^{-x}+x^2+2\right].$
D. $C_1\sin x+C_2\cos x-\left[\frac{1}{2}e^{-x}+x^2+2\right].$

35. The value of k such that the lines $\frac{x-1}{k}=\frac{y-1}{4}=\frac{z-2}{3}$ and $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z}{3}$ are coplanar is

A. -1. B. 1. C. -2. D. 2.

36. Consider a plane which is at a distance p from the origin $O = (0,0,0)$. Let $A,\ B,\ C$ be the points of intersection of that plane with the co-ordinate axis. The locus of the center of the sphere passing through $O,\ A,\ B$ and C is

A. $\frac{2}{x^2} + \frac{2}{y^2} + \frac{2}{z^2} = \frac{1}{p^2}$. B. $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{4}{p^2}$.
C. $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{2}{p^2}$. D. $\frac{4}{x^2} + \frac{4}{y^2} + \frac{4}{z^2} = \frac{1}{p^2}$.

37. Consider the circle C which is the intersection of the sphere $x^2 + y^2 + z^2 - x - y - z = 0$ and the plane $x + y + z = 1$. The radius of the sphere with center at the origin, containing the circle C is

A. 1. B. 2. C. 3. D. 4.

38. Which of the following statements are true?

A. All groups of order 4 are abelian.
B. All groups of order 6 are abelian.
C. $73^{12} - 1$ is divisible by 7.
D. A subgroup of a cyclic group must be cyclic.

39. Consider the quotient group $G = \frac{\mathbb{Q}}{\mathbb{Z}}$ under addition. Which of the following statements about G are true?

A. G is a finite group.
B. In G every element has a finite order.
C. G has no nontrivial proper subgroups.
D. G is NOT a cyclic group.

40. Let $\mathcal{G} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q} - \{0\},\ b \in \mathbb{Q} \right\}$, $\mathcal{U} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Q} \right\}$,
$\mathcal{D} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q} - \{0\} \right\}$.

Which of the following statements are true?

A. $\mathcal{G},\ \mathcal{U},\ \mathcal{D}$ are all groups under multiplication.
B. $\mathcal{D}$ is a normal subgroup of $\mathcal{G}$.
C. $\mathcal{U}$ is a normal subgroup of $\mathcal{G}$.
D. For every matrix $A \in \mathcal{U}$, $A\mathcal{D}A^{-1} \subseteq \mathcal{D}$.

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41. Let $X = \{1,2,3,4,5\}$, $\mathscr{P}(X)$ be the power set of X. Consider the ring $R = (\mathscr{P}(X), \Delta, \cap)$, for subsets $A \& B$ of X, $A \Delta B = (A \cup B) - (A \cap B)$. Which of the following statements are true about R?

A. R is a commutative ring with unity.
B. R is a field.
C. Every element in R has 'additive' order 2.
D. Every element in R has 'multiplicative' order 2.

42. Consider the ring $R = (\mathbb{Z}_{60}, +, .)$. Which of the following statements are true about R?

A. There are no maximal ideals in R.
B. There are three maximal ideals in R.
C. There are ten nonzero proper ideals in R.
D. All nonzero ideals in R are maximal.

43. Consider the group $\mathbb{Z}$ under addition $+$. Define the binary operation $*$ on $\mathbb{Z}$ by $a * b = 0$, $\forall a, b \in \mathbb{Z}$. Which of the following statements are true about R?

A. $(\mathbb{Z}, +, .)$ is a commutative ring with unity.
B. $(\mathbb{Z}, +, .)$ is a ring.
C. Every additive subgroup of $\mathbb{Z}$ is an ideal.
D. The only ideals in $\mathbb{Z}$ are of the form $n\mathbb{Z} = \{nx | x \in \mathbb{Z}\}$.

44. Let A be a nonsingular 3×3 matrix with real entries. For every nonzero eigenvalue λ of A,

A. λ is an eigenvalue of both $P^{-1}AP$, PAP^{-1} where $\det(P) \neq 0$.
B. $1 + \lambda$ is an eigenvalue of $I + A$.
C. if $\det(A) < 1$ then $|\lambda| < 1$.
D. if μ is an eigenvalue of A^{-1} then $\mu\lambda = 1$.

45. Let A be a 2×2 real matrix. Let the sum of the entries in each row of A be equal to 2. Which of the following statements is true?

A. 0 is always an eigenvalue of A.
B. 0 and 2 are always eigenvalues of A.
C. 2 is always an eigenvalue of A.
D. None of the above.

46. Let A, B be a 4×4 matrices. Denote rank of a matrix A, B by $\rho(A)$, $\rho(B)$ and adjoint of A by $\text{adj}(A)$. Which of the following statements are true?

A. $\rho(A + B) \leq \rho(A) + \rho(B)$.
B. $\rho(A - B) \leq \rho(A) - \rho(B)$.
C. $\rho(AB) \leq \rho(A)\rho(B)$.
D. If $\rho(A) = 2$ then $\text{adj}(A) = O_{4\times4}$.

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47. Consider the vector space $V = \mathbb{R}^3(\mathbb{R})$ and $B = \{v_1, v_2\} \subset V,\ 0 \notin B$. Which of the following statements are true?

A. If B is a linearly dependent set then $\exists(\alpha_1, \alpha_2) \neq (0,0)$ such that $\alpha_1 v_1 + \alpha_2 v_2 = 0$.

B. If B is a linearly dependent set then $\exists(\alpha_1, \alpha_2)$ such that $\alpha_1 \neq 0,\ \alpha_2 \neq 0$ and $\alpha_1 v_1 + \alpha_2 v_2 = 0$.

C. If B is linearly independent then $\exists$ no nonzero 2-tuple (α_1, α_2) such that $\alpha_1 v_1 + \alpha_2 v_2 \neq 0$.

D. If B is linearly independent then $\exists$ no nonzero 2-tupple (α_1, α_2) such that $\alpha_1 v_1 + \alpha_2 v_2 = 0$.

48. Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\bar{f} : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3$ be given by $f(x, y, z) = \dfrac{\bar{r}}{|\bar{r}|^n}$. The value of n for which $\text{div}(\bar{f}) = 0$ is

A. 1. B. 2. C. 3. D. 4.

49. Let R be a region in the $xy-$plane. The boundary of R is a smooth simple closed curve C which is parametrized by $C = (x(t), y(t)),\ t \in [0, 1]$. The area of R is NOT equal to

A. $\int_0^1 x(t)y'(t)dt$.

B. $-\int_0^1 y(t)x'(t)dt$.

C. $\frac{1}{2}\int_0^1 \big(x(t)y'(t) + y(t)x'(t)\big)dt$.

D. $\frac{3}{4}\int_0^1 x(t)y'(t)dt - \frac{1}{4}\int_0^1 y(t)x'(t)dt$.

50. A storage depot contains 10 machines 4 of which are defective. If a company selects 5 of these machines randomly, then what is the probability that at least 4 of the machines are NON DEFECTIVE?

A. $\frac{11}{42}$. B. $\frac{5}{21}$. C. $\frac{1}{252}$. D. none of the above.