MAY 2011 **P/ID 37451/PMAA**

Time : Three hours Maximum : 100 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

Each question carries 2 marks.

1. Define the normalizer of a in G.

2. Define cyclic R–module.

3. Define the index of nilpotence of T.

4. Define elementary divisors of T.

5. Define unitary transformation.

6. Define an algebraic number.

7. If $a \in K$ is a root of $p(x) \in F[x]$, where $F \subset K$, prove that $(x-a) | p(x)$, in $K[x]$.

8. Define group of automorphisms of K relative to F.

9. State Frobenius theorem.

10. State Lagrange's identity.

SECTION B — (5 × 6 = 30 marks)

Answer ALL questions.

Each question carries 6 marks.

11. (a) Prove that $n(k)=1+p+...+p^{k-1}$.

Or

(b) Suppose that G is the internal direct product of $N_1, N_2,....,N_k$. Prove that for $i \neq j$, $N_i \cap N_j=(e)$, and if $a \in N_i, b \in N_j$, then $ab=ba$.

12. (a) If V is an n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, prove that T satisfies a polynomial of degree n over F.

Or

(b) If $u \in V_1$ is such that $uT^{k-1}=0$, where $0<k<n_1$, prove that $u=u_0T^k$, for some $u_0 \in V_1$.

13. (a) If T is Hermitian and $vT^k=0$ for $k \geq 1$, prove that $vT=0$.

Or

(b) If $T \in A(V)$ is such that $(vT, v)=0$, for all $v \in V$, prove that $T=0$.

14. (a) Prove that the fixed field of G is a subfield of K.

Or

(b) Let K be the splitting field of $f(x)$ in $F[x]$ and let $p(x)$ be an irreducible factor of $f(x)$ in $F[x]$. If the roots of $p(x)$ are $\alpha_1, \alpha_2, \ldots. \alpha_r$, prove that for each i, there exists an automorphism σ_1 in $G(K, F)$ such that $\sigma_i(\alpha_1) = \alpha_i$.

15. (a) Prove that the general polynomial of degree $n \geq 3$ is not solvable by radicals.

Or

(b) For every prime number p and every positive integer m, prove that there exists a field having p^m elements.

SECTION C — (5 × 10 = 50 marks)

Answer ALL questions.

Each question carries 10 marks.

16. (a) State and prove Cauchy's theorem.

Or

(b) Prove that every finite abelian group is the direct product of cyclic groups.

17. (a) Prove that two nilpotent transformations are similar if and only if they have the same invariants.

Or

(b) For each $i=1, 2,..,k$ prove that $V_i \neq 0$; $V=V_1 \oplus V_2 \oplus ... \oplus V_k$ and the minimal polynomial of T_i is $q_i(x)^{l_i}$.

18. (a) Prove that the number e is transcendental.

Or

(b) State and prove Sylvester's law.

19. (a) Prove that any splitting fields E and E' of the polynomials $f(x) \in F[x]$ and $f'(t) \in F'[t]$, respectively, are isomorphic by an isomorphism ϕ with the property that $\alpha\phi = \alpha'$.

Or

(b) If F is of characteristic 0 and if a, b are algebraic over F, prove that there exists an element $e \in F(a,b)$ such that $F(a,b) = F(e)$.

20. (a) State and prove Wedderburn's theorem.

Or

(b) State and prove Left Division algorithm.