PART A

Instructions:

- There are 13 questions in this part. Each question carries 4 marks. Answer all questions.

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1. For a polynomial $f(x)$, let $f^{(n)}(x)$ denote the n^{th}–derivative for $n \geq 1$ and $f^{(0)}(x) = f(x)$. Is the following true or false ? Give brief reasons.

   $$f^{(n)}(a) = 0, for\ n = 0, 1, \ldots, k \iff (x{-}a)^{(k+1)}\ divides\ f(x)$$

2. (a) Find the value of $\sum_{k=1}^{n} k \cdot k!$ as a function of n.

   (b) True or false ? Give brief reasons.

   $$\sum_{k=2}^{1000} \frac{1}{log_k N} = \frac{1}{log_{1000!} N}$$

3. Let $p(x)$ be a polynomial such that when divided by $(x-1)$ it leaves the remainder 2 and when divided by $(x-2)$ it leaves the remainder 1. What is the remainder when it is divided by $(x-1)(x-2)$?

4. Prove that for any fixed $n > 2$, among the two integers $2^n - 1$ and $2^n + 1$, at the most one of them can be a prime.

5. Show that $n! \leq (\frac{n+1}{2})^n$, $\forall n \geq 1$.

6. Show that the sum

   $$\sum_{k=1}^{n} k! \neq m^2$$

   for any integer m, for $n \geq 4$.

7. Give an example of a subset S of the plane such that the following property is satisfied:

   There exist infinitely many lines ℓ in the plane such that the intersection $\ell \cap S$ is a line segment of any length however large.

8. True or false ? Give brief reasons. *There exist non-zero ploynomials $f(x)$ with integer coefficients of any degree $d \geq 5$ which vanish at $x = 2, 1 + \sqrt{-3}$ and $1 + \sqrt{-5}$.*

9. If the sum of 113 terms of an arithmetic progression is equal to 6780, then find the 57th term of the arithmetic progression. Give an example of such an arithmetic progression.

10. Show that if p and $p + 2$ with $p \geq 5$, are both primes then the number $p + 1$ is always divisible by 6.

11. Show that for all real numbers a, b, c,

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$$

    Further show that 3 is the smallest real number with this property.

12. True or false ? Give brief reasons. *If a, b are positive integers then the number*

$$(a + \sqrt{b})^n + (a - \sqrt{b})^n$$

    is an integer for all values of n.

13. True or false ? Give brief reasons. *The number of common tangents to the circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 4x + 3 = 0$ is 1.*

Part B

Instructions:

- There are 6 questions in this part. Each question carries 8 marks. Answer all questions.

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1. Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ form a triangle with circumcentre $(0, 0)$. Show that ABC is an equilateral triangle if and only if

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$$

2. Show the following:
   (a) $\lim_{x\to 0} \sin(x)\log(x) = 0$
   (b) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$
   (c) $\lim_{x\to\infty}(cos(\frac{\alpha}{x}))^x = 1$, where α is a constant.

3. In a jail with 100 rooms, all locked initially, 100 rioters break in and disturb the rooms in the following way. First one stops at all rooms and opens them all. Second rioters stops at rooms numbered $2, 4, 6, \ldots$ and locks open rooms, and leaving the other rooms as they were. The third rioter stops at rooms numbered $3, 6, 9, \ldots$ and again opens a locked room and locks an open room, leaving others undisturbed. And this process continues. After all the 100 rioters have left which rooms would be open?

4. (a) If two sides of a triangle are given, then show that the area of this triangle is maximum if the sides are perpendicular to each other.

(b) Construction: For any given positive real number λ, find a point D on a line AB such that the difference $AD^2 - BD^2 = \lambda$. (Note that the point D need not be within the segment AB).

5. Show that if $[x]$ denote the greatest integer $\leq x$, then

$$\binom{n}{3} - [\frac{n}{3}]$$

is a natural number divisible by 3 for all values of n.

6. Is it possible to remove one square from a 5×5 board so that the remaining 24 squares can be covered by eight 3×1 rectangles? If yes, find all such squares.

(Hint: A domino is a 2×1 rectangle. As you may know, if two diagonally opposite squares of an ordinary 8×8–chessboard are removed, the remaining 62 squares cannot be covered by 31 non-overlapping dominos. The reason being, after removing the two corners 32 squares of one colour and 30 of the other are left. No matter how you place a domino it will cover one white and one black square.)