26/7 9.30 pm

12/1

2005

MATHEMATICS

Paper 1

Time : 3 Hours] *[Maximum Marks : 300*

INSTRUCTIONS

*Candidates should attempt **all** the questions in Parts A, B & C. However, they have to choose only **three** questions in Part D. The number of marks carried by each question is indicated at the end of the question.*

Answers must be written in English.

This paper has four parts :

A	20 marks
B	100 marks
C	90 marks
D	90 marks

Marks allotted to each question are indicated in each part.

PART A

4×5

Each question carries 5 marks.

1. (a) Let F(x) be the vector space of all polynomials over a field F. Prove that F(x) is not a finite dimensional space.

   (b) Let $\mathbb{R}$ be the set of real numbers and $a \in \mathbb{R}$. Give an example of a function of $\mathbb{R}$ into $\mathbb{R}$ which is continuous at a and not continuous at all other points of $\mathbb{R}$.

   (c) A particle is moving along a curve given by

   $$x = a \cos \theta, \quad y = a \sin \theta \quad \text{and} \quad z = a\, \theta \tan \alpha.$$

   Find the radius of curvature and the torsion.

   (d) If a solid displaces $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ of its volume respectively when it floats in three different liquids, then find the volume it displaces when it floats in a mixture formed of equal volumes of liquids.

PART B

10×10=100

Each question carries 10 marks.

1. If A is the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, find the ranks of A, A^2 and A^3.

2. Prove that $\sin x < x < \tan x$ for all $0 < x < \pi/2$.

3. Prove that the equation

$$2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$$

represents a pair of planes and find the angle between them.

4. Solve $2xy\, dy - (x^2 + y^2 + 1)\, dx = 0$.

5. If $\bar{a}$ is a constant vector, show that curl $\frac{\bar{a} \times \bar{r}}{r^3} = \frac{-\bar{a}}{r^3} + \frac{3\bar{r}}{r^5} (\bar{a} \cdot \bar{r})$.

6. Define the concept of contraction of tensors and show that the inner product of the tensors A_r^p and B_t^{q-s} is a tensor of rank three.

7. State and prove Lami's theorem on coplanar forces acting at a point which are in equilibrium.

8. A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, then show that the equation of its path is $r \cos \frac{\theta}{\sqrt{2}} = a$.

9. A uniform rod of length 2a can turn freely about one end which is fixed at a height h (< 2a) above the surface of a liquid. If the densities of the rod and the liquid are ρ and σ respectively, then show that the rod can rest either in a vertical position or inclined at an angle θ to the vertical such that $\cos\theta = \frac{h}{2a}\sqrt{\frac{\sigma}{\sigma - \rho}}$.

10. Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

PART C

$6 \times 15 = 90$

Each question carries 15 marks.

1. State and prove the Cayley – Hamilton theorem and, using this, find the inverse of the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

2. State and prove the Lagrange's mean-value theorem and, using this, prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$.

3. (a) Obtain the divergence and curl of a vector in spherical polar coordinates.

   (b) Define the Christoffel symbols of the first and second kind. If $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2$, find the values of $[22, 1]$ and $\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\}$.

4. State the Serret – Frenet formulae. Given the space curve $x = t$, $y = t^2$ and $z = \frac{2}{3}t^3$, find the values of k and τ at $t = 1$.

5. (a) State the principle of virtual work.

   (b) Five weightless rods of equal length are joined together so as to form a rhombus ABCD with one diagonal BD. If a weight w be attached to C and the system be suspended from A, show that there is a thrust in BD equal to $\frac{w}{\sqrt{3}}$.

6. (a) A square is placed in a liquid with one side in the [illegible] how to draw a horizontal line in the square dividing it into [illegible] portions, the thrusts on which are the same.

(b) A rectangle is immersed vertically in a homogeneous liquid with two of its sides horizontal and at depths α and β below the surface. Show that the depth of centre of pressure is $\frac{2(\alpha^2 + \alpha\beta + \beta^2)}{3(\alpha + \beta)}$.

PART D

3×30=90

*Answer any **three** of the following questions. Each question carries 30 marks.*

1. (a) Prove that a real quadratic form X′AX is positive definite if and only if the leading principal minors of A are all positive.

(b) The matrix of a quadratic form is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Examine whether the quadratic form is definite or not.

2. (a) Let $f(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and let $f(0, 0) = 0$.

Then show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

(b) If $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ and $(a, b) \in D$ such that f_x exists in a neighbourhood of (a, b) and f_{xy} is a continuous at (a, b), then prove that f_{yx} exists and $f_{yx} = f_{xy}$ at (a, b).

(c) In (b), if f_x and f_y are both derivable at (a, b), then prove that $f_{xy} = f_{yx}$ at (a, b).

3. State the Gauss divergence theorem and verify this theorem for $\vec{A} = 4x\,\bar{i} - 2y^2\,\bar{j} + z^2\,\bar{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

4. Solve the differential equations :

(a) $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$

(b) $(D^2 + 2D + 1)\,y = x \cos x$, where $D = \dfrac{d}{dx}$

5. (a) A ladder rests at an angle α to the horizon with its ends resting on a smooth floor and against a smooth vertical wall the lower end being attached by a string to the junction of the wall and the floor. Find the tension of the string. Find also the tension of the string when a man, whose weight is one-half that of the ladder, has descended the ladder two thirds of its length.

(b) A smooth wire bent in the form of a parabola is fixed with its axis verticle and vertex downwards. A particle of mass m oscillates on the wire coming to rest at the extremities of the latus rectum. Show that the reaction of the wire on the particle when passing through the vertex is 2 mg.

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28/7/05
3 to 6 pm

2005

MATHEMATICS

Paper 2

Time : 3 Hours] *[Maximum Marks : 300*

INSTRUCTIONS

Candidates should attempt ***all*** *the questions in Parts A, B & C. However, they have to choose only* ***three*** *questions in Part D. The number of marks carried by each question is indicated at the end of the question.*

Answers must be written in English.

This paper has four parts :

A	20 marks
B	100 marks
C	90 marks
D	90 marks

Marks allotted to each question are indicated in each part.

PART A

Each question carries 5 marks.

1. (a) Let H be a sub-group of the symmetric group S_n of order n. Prove that either all the permutations in H are even or exactly half of them are even.

(b) Give an example of a continuous function which is not uniformly continuous.

(c) Evaluate $\int_0^1 \frac{1}{1+x}\,dx$ using the Simpson's rule with 8 equal sub-intervals. Compare your value with the exact value of the integral.

(d) State the assumptions under which a Poisson distribution is approximated from a binomial distribution. A car-hire firm has two cars which it hires day by day. The demand for a car on each day follows a Poisson distribution with mean 1·5. Find the proportion of days on which neither car is used and the proportion of days on which demand is refused ($e^{-1 \cdot 5} = 0 \cdot 2231$).

PART B

10×10=100

Each question carries 10 marks.

1. Prove that any group of order 391 is cyclic.

2. Let A be a compact set in a metric space. Prove that every continuous function defined from A into any metric space is uniformly continuous.

3. State and prove Liouville's theorem on analytic functions.

4. Find a complete integral of the equation

$$p^2x + q^2y = z, \text{ where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}$$

5. Derive the Newton – Raphson interation formula for finding the best approximate root of a given transcendental/polynomial equation $f(x) = 0$. Find a real root of the equation $x = e^{-x}$ using this formula. Perform 4 interations.

6. Determine the constants a and b by the method of least squares such that the curve $y = ae^{bx}$ fits with the following data.

x	2	4	6	8	10
y	4·077	11·084	30·128	81·897	222·62

7. Define a two-person zero-sum game. Explain briefly the theory of dominance in the solution of rectangular games. Solve the following game whose pay-off matrix is given below (the pay-off is for player A).

Player B

$$\text{Player A} \begin{bmatrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{bmatrix}$$

8. A book binder has one printing press, one binding machine and the manuscripts of a number of books. Time required to perform the printing and binding operations for each book are given below.

Book	1	2	3	4	5	6
Printing time (hrs)	30	120	50	20	90	110
Binding time (hrs)	80	100	90	60	30	10

Determine the sequence in which the jobs (books) should be processed in order to minimise the total time required to process all the books. Determine the idle time for both the machines.

9. (a) Show that the generalised momentum conjugate to a cyclic coordinate is conserved.

(b) A particle of mass m is projected with initial velocity u at an angle α with the horizontal using the Lagrange's equation of motion. Describe the motion of the projectile. Neglect the air resistance.

10. (a) Determine whether the motion specified by

$$\vec{v} = \frac{k(x\vec{j} - y\vec{i})}{x^2 + y^2}, \; k = \text{constant} \neq 0$$

is a possible motion for an incompressible fluid. If so determine the equations of stream lines. Also show that the motion is of potential kind. Find the velocity potential.

(b) What arrangements of sources and sinks will give rise to the function

$$w = \log\left(z - \frac{a^2}{z}\right)$$

PART C

6×15=90

Each question carries 15 marks.

1. Prove that every principal ideal domain is a unique factorization domain. Is the converse true ? Justify your answer.

2. (a) Prove that any metric space x can be imbedded in a complete metric space containing x as a dense subspace.

   (b) Define the concept of the completion of a metric space and prove its existence and uniqueness.

3. (a) Solve the equation $r + (a + b)s + abt = xy$, where r, s, t denote $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ respectively.

   (b) Find the surface passing through the two lines $x = z = 0$ and $z - 1 = x - y = 0$ and satisfying the equation $r - 4s + 4t = 0$.

4. Fit the following data by cubic spline curve and find f(0·5) and f(2·5). Use natural spline.

x	0	1	2	3	4
f(x)	— 8	— 7	0	19	56

5. (a) A can hit the target four times in five shots, B three times in four shots and c twice in three shots. Calculate the probability that

   (i) A, B, C all may hit, and

   (ii) B, C may hit and A may lose

   (b) An item is manufactured by three machines M_1, M_2 and M_3. Out of the goods manufactured in a specified period, 50% are manufactured on M_1, 30% on M_2 and the remaining on M_3. It is also known that 2% of items produced by M_1 and M_2 are defective while 3% by M_3 are defective. All the items are put into one bin and an item drawn at random was found to be defective. Find the probability that it was made on M_1.

6. A uniform sphere rolls down an inclined plane, rough enough to prevent any sliding. Discuss the motion.

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3×30=90

PART D

Answer any ***three*** *of the following questions. Each question carries 30 marks.*

1. Let G be a finite group, p a prime number and r a positive integer such that p^r divides the order of G and p^{r+1} doesn't divide the order of G. Prove the following, where n is the number of sub-groups of order p^r in G.

   (a) $n > 0$

   (b) $n \equiv 1 \pmod p$

   (c) Any two sub-groups of order p^r are conjugate to each other.

2. (a) If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge to A and B respectively, $\sum_{n=0}^{\infty} a_n$ converges absolutely and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ (for $n = 0, 1, 2, \ldots$), then prove that $\sum_{n=0}^{\infty} c_n = AB$.

   (b) Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose $-\infty \le \alpha \le \beta \le \infty$. Then prove that there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that $\liminf_{n \to \infty} s'_n = \alpha$ and $\limsup_{n \to \infty} s'_n = \beta$.

3. (a) If n, the number of arrivals in a queue in time t follows the Poisson distribution

   $$p_n(t) = e^{-\lambda t}(\lambda t)^n / n!,$$

   then prove that the inter arrival time T obeys the negative exponential law.

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(b) The rate of arrivals of customers at a public telephone booth follows Poisson distribution with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution with a mean time of three minutes.

(i) What is the probability that a person arriving at a booth will have to wait ?

(ii) What is the average length of the non-empty queue that forms from time to time ?

(iii) The authorities install a second booth when they are convinced that the customers would expect waiting for atleast 3 minutes for their turn to make a call. By how much time should the flow of customers increase in order to justify a second booth ?

4. Consider the following transport problem :

Factory	Godowns						Availability
	1	2	3	4	5	6	
A	7	5	7	7	5	3	60
B	9	11	6	11	—	5	20
C	11	10	6	2	2	2	90
D	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	

It is not possible to transport any quantity from factory B to godown 5. Determine the following :

(a) Initial feasible solution by Vogel's approximation method.

(b) Optimum basic feasible solution.

(c) Is the optimum solution unique ? Justify your answer.

5. (a) Show that $\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1$ is a possible form for the bounding surface of a liquid and find an expression for the normal velocity.

(b) A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\phi(vt - x)$. Prove that the velocity u (at time t and distance x from the origin) is being given by

$$v + \frac{(u_0 - v)\,\phi(vt)}{\phi(vt - x)}.$$

(c) Show that, in the case of a two dimensional motion, there always exists a stream function whether the motion is irrotational or not.