Mathematics Paper
2007

IMPORTANT NOTE FOR CANDIDATES

- **Attempt ALL the 29 questions.**
- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-29 (subjective questions) carry *fifteen* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

1. Which of the following sets is a basis for the subspace

$$W = \left\{ \begin{bmatrix} x & y \\ 0 & t \end{bmatrix} : x + 2y + t = 0,\ y + t = 0 \right\}$$

of the vector space of all real 2×2 matrices?

(A) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \right\}$

(D) $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

2. Let G be an Abelian group of order 10. Let $S = \{g \in G : g^{-1} = g\}$. Then the number of non-identity elements in S is

(A) 5
(B) 2
(C) 1
(D) 0

3. Let R be the ring of polynomials over $\mathbf{Z}_2$ and let I be the ideal of R generated by the polynomial $x^3 + x + 1$. Then the number of elements in the quotient ring R/I is

(A) 2
(B) 4
(C) 8
(D) 16

4. Let $f:\mathbf{R}\to\mathbf{R}$ be a continuous function. If $\int_0^x f(2t)\,dt=\frac{x}{\pi}\sin(\pi x)$ for all $x\in\mathbf{R}$, then $f(2)$ is equal to

(A) -1
(B) 0
(C) 1
(D) 2

5. Suppose (c_n) is a sequence of real numbers such that $\lim_{n\to\infty}|c_n|^{1/n}$ exists and is non-zero. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is equal to r, then the radius of convergence of the power series $\sum_{n=1}^{\infty} n^2 c_n x^n$ is

(A) less than r
(B) greater than r
(C) equal to r
(D) equal to 0

6. The rank of the matrix $\begin{bmatrix} 1 & 4 & 8 \\ 2 & 10 & 22 \\ 0 & 4 & 12 \end{bmatrix}$ is

(A) 3
(B) 2
(C) 1
(D) 0

7. If k is a constant such that $xy+k=e^{(x-1)^2/2}$ satisfies the differential equation

$$x\frac{dy}{dx}=(x^2-x-1)y+(x-1),$$ then k is equal to

(A) 1

(B) 0

(C) –1

(D) –2

8. Which of the following functions is uniformly continuous on the domain as stated?

(A) $f(x)=x^2, \quad x\in \mathbf{R}$

(B) $f(x)=\dfrac{1}{x}, \quad x\in [1,\infty)$

(C) $f(x)=\tan x, \quad x\in (-\pi/2, \pi/2)$

(D) $f(x)=[x], \quad x\in [0,1]$

($[x]$ is the greatest integer less than or equal to x)

9. Let $A(t)$ denote the area bounded by the curve $y=e^{-|x|}$, the x-axis and the straight lines $x=-t$ and $x=t$. Then $\lim_{t\to\infty} A(t)$ is equal to

(A) 2

(B) 1

(C) 1/2

(D) 0

10. Let C denote the boundary of the semi-circular disk $D=\{(x,y)\in\mathbf{R}^2 : x^2+y^2\leq 1,\ y\geq 0\}$ and let $\varphi(x,y)=x^2+y$ for $(x,y)\in D$. If $\hat{n}$ is the outward unit normal to C, then the integral $\oint_C (\vec{\nabla}\varphi)\cdot\hat{n}\,ds$, evaluated counter-clockwise over C, is equal to

(A) 0

(B) $\pi-2$

(C) π

(D) $\pi+2$

11. Let $\vec{u}=(a e^x \sin y-4x)\hat{\mathrm{i}}+(2y+e^x\cos y)\hat{\mathrm{j}}+az\,\hat{\mathrm{k}}$, where a is a constant. If the line integral $\oint_C \vec{u}\cdot d\vec{r}$ over every closed curve C is zero, then a is equal to

(A) -2

(B) -1

(C) 0

(D) 1

12. One of the integrating factors of the differential equation $(y^2-3xy)dx+(x^2-xy)dy=0$ is

(A) $1/(x^2y^2)$

(B) $1/(x^2y)$

(C) $1/(xy^2)$

(D) $1/(xy)$

13. Let $f:\mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Which of the following statements holds regarding the continuity and the existence of partial derivatives of f at $(0,0)$?

(A) Both partial derivatives of f exist at $(0,0)$ and f is continuous at $(0,0)$

(B) Both partial derivatives of f exist at $(0,0)$ and f is NOT continuous at $(0,0)$

(C) One partial derivative of f does NOT exist at $(0,0)$ and f is continuous at $(0,0)$

(D) One partial derivative of f does NOT exist at $(0,0)$ and f is NOT continuous at $(0,0)$

14. Let (a_n) be an increasing sequence of positive real numbers such that the series $\sum_{k=1}^{\infty} a_k$ is divergent. Let $s_n=\sum_{k=1}^{n} a_k$ for $n=1,2,\ldots$ and $t_n=\sum_{k=2}^{n}\frac{a_k}{s_{k-1}s_k}$ for $n=2,3,\ldots$. Then $\lim_{n\to\infty} t_n$ is equal to

(A) $1/a_1$

(B) 0

(C) $1/(a_1+a_2)$

(D) a_1+a_2

15. For every function $f:[0,1]\to\mathbf{R}$ which is twice differentiable and satisfies $f'(x)\geq 1$ for all $x\in[0,1]$, we must have

(A) $f''(x)\geq 0$ for all $x\in[0,1]$

(B) $f(x)\geq x$ for all $x\in[0,1]$

(C) $f(x_2)-x_2\leq f(x_1)-x_1$ for all $x_1,x_2\in[0,1]$ with $x_2\geq x_1$

(D) $f(x_2)-x_2\geq f(x_1)-x_1$ for all $x_1,x_2\in[0,1]$ with $x_2\geq x_1$

16. (a) Let $M = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$. Determine the eigenvalues of the matrix $B = M^2 - 2M + I$. (9)

(b) Let N be a square matrix of order 2. If the determinant of N is equal to 9 and the sum of the diagonal entries of N is equal to 10, then determine the eigenvalues of N. (6)

17. (a) Using the method of variation of parameters, solve the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2,$$

given that x and $\frac{1}{x}$ are two solutions of the corresponding homogeneous equation.

(9

(b) Find the real number α such that the differential equation

$$\frac{d^2y}{dx^2}+2(\alpha-1)(\alpha-3)\frac{dy}{dx}+(\alpha-2)y=0$$

has a solution $y(x)=a\cos(\beta x)+b\sin(\beta x)$ for some non-zero real numbers a, b, β. (6

18. (a) Let a, b, c be non-zero real numbers such that $(a-b)^2 = 4ac$. Solve the differential equation $a\left(x+\sqrt{2}\right)^2 \frac{d^2y}{dx^2} + b\left(x+\sqrt{2}\right)\frac{dy}{dx} + cy = 0$. (9)

(b) Solve the differential equation

$dx + (e^{y \sin y} - x)(y \cos y + \sin y)dy = 0$. (6)

19. Let $f(x, y)=x(x-2y^2)$ for $(x, y) \in \mathbf{R}^2$. Show that f has a local minimum at $(0, 0)$ on every straight line through $(0, 0)$. Is $(0, 0)$ a critical point of f? Find the discriminant of f at $(0, 0)$. Does f have a local minimum at $(0, 0)$? Justify your answers. (15)

20. (a) Find the finite volume enclosed by the paraboloids $z=2-x^2-y^2$ and $z=x^2+y^2$. (9)

(b) Let $f:[0,3]\to\mathbf{R}$ be a continuous function with $\int_0^3 f(x)dx=3$. Evaluate

$$\int_0^3 [x\,f(x)+\int_0^x f(t)dt]\,dx\,.$$ (6)

21. (a) Let S be the surface $\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+2z=2, z\geq 0\}$, and let $\hat{n}$ be the outward unit normal to S. If $\vec{F}=y\,\hat{\mathrm{i}}+xz\,\hat{\mathrm{j}}+(x^2+y^2)\,\hat{\mathrm{k}}$, then evaluate the integral $\iint_S \vec{F}\cdot\hat{n}\,dS$. (9)

(b) Let $\vec{r}=x\,\hat{\mathrm{i}}+y\,\hat{\mathrm{j}}+z\,\hat{\mathrm{k}}$ and $r=|\vec{r}\,|$. If a scalar field φ and a vector field $\vec{u}$ satisfy $\vec{\nabla}\varphi=\vec{\nabla}\times\vec{u}+f(r)\,\vec{r}$, where f is an arbitrary differentiable function, then show that $\nabla^2\varphi=r\,f'(r)+3f(r)$. (6)

22. (a) Let D be the region bounded by the concentric spheres $S_1 : x^2 + y^2 + z^2 = a^2$ and $S_2 : x^2 + y^2 + z^2 = b^2$, where $a < b$. Let $\hat{n}$ be the unit normal to S_1 directed away from the origin. If $\nabla^2 \varphi = 0$ in D and $\varphi = 0$ on S_2, then show that

$$\iiint_D \left| \vec{\nabla} \varphi \right|^2 dV + \iint_{S_1} \varphi \, (\vec{\nabla} \varphi) \cdot \hat{n} \, dS = 0. \qquad (9)$$

(b) Let C be the curve in $\mathbf{R}^3$ given by $x^2 + y^2 = a^2$, $z = 0$ traced counter-clockwise, and let $\vec{F} = x^2 y^3 \, \hat{\mathrm{i}} + \hat{\mathrm{j}} + z \, \hat{\mathrm{k}}$. Using Stokes' theorem, evaluate $\oint_C \vec{F} \cdot d\vec{r}$. (6)

23. Let V be the subspace of $\mathbf{R}^4$ spanned by the vectors (1,0,1,2), (2,1,3,4) and (3,1,4,6). Let $T: V \to \mathbf{R}^2$ be a linear transformation given by $T(x, y, z, t) = (x - y,\ z - t)$ for all $(x, y, z, t) \in V$. Find a basis for the null space of T and also a basis for the range space of T.

(15)

24. (a) Compute the double integral $\iint\limits_{D}(x+2y)\,dx\,dy$, where D is the region in the xy-plane bounded by the straight lines $y=x+3$, $y=x-3$, $y=-2x+4$ and $y=-2x-2$. (9)

(b) Evaluate $\int\limits_{0}^{\pi/2}\left[\int\limits_{\pi/2}^{\pi}\frac{\sin x}{x}dx\right]dy+\int\limits_{\pi/2}^{\pi}\left[\int\limits_{y}^{\pi}\frac{\sin x}{x}dx\right]dy$. (6)

25. (a) Does the series $\sum_{k=1}^{\infty} \frac{(-1)^k k + x^k}{k^2}$ converge uniformly for $x \in [-1, 1]$? Justify. (9)

(b) Suppose (f_n) is a sequence of real-valued functions defined on $\mathbf{R}$ and f is a real-valued function defined on $\mathbf{R}$ such that $|f_n(x) - f(x)| \le |a_n|$ for all $n \in \mathbf{N}$ and $a_n \to 0$ as $n \to \infty$. Must the sequence (f_n) be uniformly convergent on $\mathbf{R}$? Justify. (6)

26. (a) Suppose f is a real-valued thrice differentiable function defined on $\mathbf{R}$ such that $f'''(x)>0$ for all $x \in \mathbf{R}$. Using Taylor's formula, show that

$f(x_2)-f(x_1)>(x_2-x_1)f'\left(\frac{x_1+x_2}{2}\right)$ for all x_1 and x_2 in $\mathbf{R}$ with $x_2>x_1$. (9)

(b) Let (a_n) and (b_n) be sequences of real numbers such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbf{N}$. Must there exist a real number x such that $a_n \leq x \leq b_n$ for all $n \in \mathbf{N}$? Justify your answer. (6)

27. Let G be the group of all 2×2 matrices with real entries with respect to matrix multiplication. Let G_1 be the smallest subgroup of G containing $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and G_2 be the smallest subgroup of G containing $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Determine all elements of G_1 and find their orders. Determine all elements of G_2 and find their orders. Does there exist a one-to-one homomorphism from G_1 onto G_2? Justify. (15)

28. (a) Let p be a prime number and let $\mathbf{Z}$ be the ring of integers. If an ideal J of $\mathbf{Z}$ contains the set $p\mathbf{Z}$ properly, then show that $J = \mathbf{Z}$. (Here $p\mathbf{Z} = \{ px : x \in \mathbf{Z} \}$.) (9)

(b) Consider the ring $R = \{ a + ib : a, b \in \mathbf{Z} \}$ with usual addition and multiplication. Find all invertible elements of R. (6)

29. (a) Suppose E is a non-empty subset of $\mathbf{R}$ which is bounded above, and let $\alpha = \sup E$. If E is closed, then show that $\alpha \in E$. If E is open, then show that $\alpha \notin E$. (9)

(b) Find all limit points of the set $E = \left\{ n + \frac{1}{2m} : n, m \in \mathbf{N} \right\}$. (6)