Module 3

Quantization and Coding

Lesson 12

Logarithmic Pulse Code Modulation (Log PCM) and Companding

After reading this lesson, you will learn about:

- ***Reason for logarithmic PCM;***
- ***A-law and μ–law Companding;***

In a linear or uniform quantizer, as discussed earlier, the quantization error in the k-th sample is

$$e_k = x(t) - x_q(kT_s) \qquad 3.12.1$$

and the maximum error magnitude in a quantized sample is,

$$Max|e_k| = \frac{\delta}{2} \qquad 3.12.2$$

So, if x (t) itself is small in amplitude and such small amplitudes are more probable in the input signal than amplitudes closer to '± V', it may be guessed that the quantization noise of such an input signal will be significant compared to the power of x (t). This implies that SQNR of usually low signal will be poor and unacceptable. In a practical PCM codec, it is often desired to design the quantizer such that the SQNR is almost independent of the amplitude distribution of the analog input signal x (t).

This is achieved by using a non-uniform quantizer. A non-uniform quantizer ensures smaller quantization error for small amplitude of the input signal and relatively larger step size when the input signal amplitude is large. The transfer characteristic of a non – uniform quantizer has been shown in **Fig 3.12.1**. A non-uniform quantizer can be considered to be equivalent to an amplitude pre-distortion process [denoted by y = c (x) in **Fig 3.12.2**] followed by a uniform quantizer with a fixed step size 'δ'. We now briefly discuss about the characteristics of this pre-distortion or 'compression' function y = c (x).

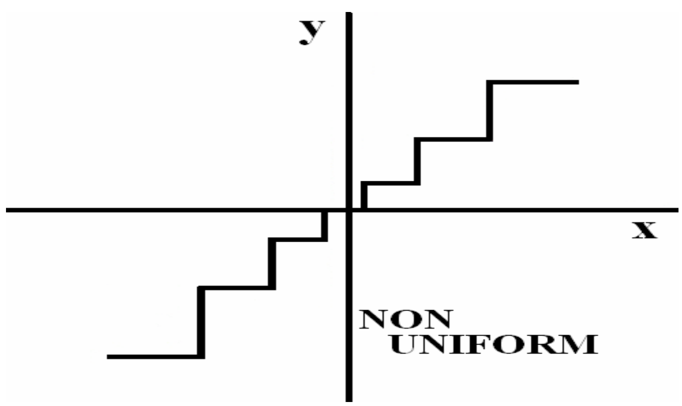

Fig 3.12.1 *Transfer characteristic of a non-uniform quantizer*

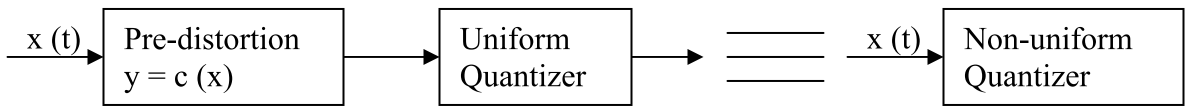

Fig. 3.12.2 *An equivalent form of a non-uniform quantizer*

Mathematically, c (x) should be a monotonically increasing function of 'x' with odd symmetry **Fig 3.12.3**. The monotonic property ensures that c^{-1} (x) exists over the range of 'x(t)' and is unique with respect to c (x) i.e., $c(x) \times c^{-1}(x) = 1$.

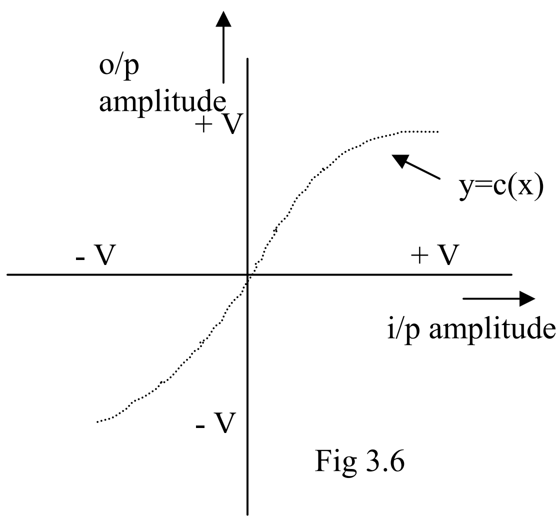

Fig. 3.12.3 *A desired transfer characteristic for non-linear quantization process*

Remember that the operation of c^{-1} (x) is necessary in the PCM decoder to get back the original signal undistorted. The property of odd symmetry i.e., c (-x) = - c (x) simply takes care of the full range '± V' of x (t). The range '± V' of x (t) further implies the following:

$$
\begin{aligned}
c(x) &= +V, \quad && \text{for } x = +V; \\
&= 0, && \text{for } x = 0; \\
&= -V, && \text{for } x = -V;
\end{aligned}
\qquad 3.12.3
$$

Let the k-th step size of the equivalent non-linear quantizer be 'δ_k' and the number of signal intervals be 'M'. Further let the k-th representation level after quantization when the input signal lies between 'x_k' and 'x_{k+1}' be 'y_k' where

$$y_k = \frac{1}{2}\left(x_k + x_{k+1}\right), \; k = 0,1,\ldots..,(M-1) \qquad 3.12.4$$

The corresponding quantization error 'e_k' is

$$e_k = x - y_k \,; \quad x_k < x \le x_{k+1}$$

Now observe from **Fig 3.12.3** that 'δ_k' should be small if '$\frac{dc(x)}{dx}$' i.e., the slope of $y = c(x)$ is large.

In view of this, let us make the following simple approximation on c (x):

$$\frac{dc(x)}{dx} \simeq \frac{2V}{M}\frac{1}{\delta_k}, \qquad k = 0,1,\ldots\ldots,(M-1) \qquad 3.12.5$$

and

$$\delta_k = x_{k+1} - x_k, \qquad k = 0,1,\ldots\ldots,(M-1)$$

Note that, '$\frac{2V}{M}$' is the fixed step size of the uniform quantizer **Fig. 3.12.2**.

Let us now assume that the input signal is zero mean and its pdf p(x) is symmetric about zero. Further for large number of intervals we may assume that in each interval I_k, $k = 0,1,\ldots,(M-1)$, the p(x) is constant. So if the input signal x (t) is between x_k and x_{k+1}, i.e.,

$x_k < x \le x_{k+1}$,

$$p(x) \simeq p(y_k)$$

So, the probability that x lies in the k-th interval I_k,

$$I_k = p_k \triangleq P_r(x_k < x \le x_{k+1}) = p(y_k)\delta_k \qquad 3.12.6$$

where, $\sum_{0}^{M-1} P_r(x_k < x \le x_{k+1}) = 1$

Now, the mean square quantization error $\overline{e^2}$ can be determined as follows:

$$\overline{e^2} = \int_{-V}^{+V} (x - y_k)^2 p(x)dx$$

$$= \sum_{k=0}^{M-1} \int_{x_k}^{x_{k+1}} (x - y_k)^2 p(y_k)dx$$

$$= \sum_{k=0}^{M-1} \frac{p_k}{\delta_k} \int_{x_k}^{x_{k+1}} (x - y_k)^2 dx$$

$$= \sum_{k=0}^{M-1} \frac{p_k}{\delta_k}\frac{1}{3}\left[(x_{k+1} - y_k)^3 - (x_k - y_k)^3\right]$$

$$= \sum_{k=0}^{M-1} \frac{1}{3}\left(\frac{p_k}{\delta_k}\right)\left\{\left[x_{k+1} - \frac{1}{2}(x_k + x_{k+1})\right]^3 - \left[x_k - \frac{1}{2}(x_k + x_{k+1})\right]^3\right\}$$

$$=\frac{1}{3}\sum_{k=0}^{M-1}\frac{p_k}{\delta_k}\frac{1}{4}\delta_k^3=\frac{1}{12}\sum_{k=0}^{M-1}p_k\delta_k^2 \qquad 3.12.7$$

Now substituting

$$\delta_k \simeq \frac{2V}{M}\left[\frac{dc(x)}{dx}\right]^{-1}$$

in the above expression, we get an approximate expression for mean square error as

$$\overline{e^2}=\frac{V^2}{3M^2}\sum_{k=0}^{M-1}p_k\left[\frac{dc(x)}{dx}\right]^{-2} \qquad 3.12.8$$

The above expression implies that the mean square error due to non-uniform quantization can be expressed in terms of the continuous variable x, -V< x < +V, and having a pdf p (x) as below:

$$\overline{e^2}\simeq\frac{V^2}{3M^2}\int_{-V}^{+V}p(x)\left[\frac{dc(x)}{dx}\right]^{-2}dx \qquad 3.12.9$$

Now, we can have an expression of SQNR for a non-uniform quantizer as:

$$SQNR\simeq\left(\frac{3M^2}{V^2}\right)\frac{\int_{-V}^{+V}x^2p(x)\,dx}{\int_{-V}^{+V}p(x)\left[\frac{dc(x)}{dx}\right]^{-2}dx} \qquad 3.12.10$$

The above expression is important as it gives a clue to the desired form of the compression function y = c(x) such that the SQNR can be made largely independent of the pdf of x (t).

It is easy to see that a desired condition is:

$$\frac{dc(x)}{dx}=\frac{K}{x}$$ where –V < x < +V and K is a positive constant.

i.e., $$c(x)=V+K\ln\left(\frac{x}{V}\right)$$ for x > 0 3.12.11

and c (x) = - c (x) 3.12.12

Note:
Let us observe that c (x) → ± ∞ as x → 0 from other side. Hence the above c(x) is not realizable in practice. Further, as stated earlier, the compression function c (x) must pass through the origin, i.e., c (x) = 0, for x = 0. This requirement is forced in a compression function in practical systems.

There are two popular standards for non-linear quantization known as
(a) The μ - law companding
(b) The A – law companding.

The μ - law has been popular in the US, Japan, Canada and a few other countries while the A - law is largely followed in Europe and most other countries, including India, adopting ITU-T standards.

The compression function c (x) for μ - law companding is (**Fig. 3.12.4** and **Fig. 3.12.5**):

$$\frac{c(|x|)}{V} = \frac{\ln\left(1+\frac{\mu|x|}{V}\right)}{\ln(1+\mu)}, \qquad 0 \le \frac{|x|}{V} \le 1.0 \qquad 3.12.13$$

'μ' is a constant here. The typical value of μ lies between 0 and 255. μ = 0 corresponds to linear quantization.

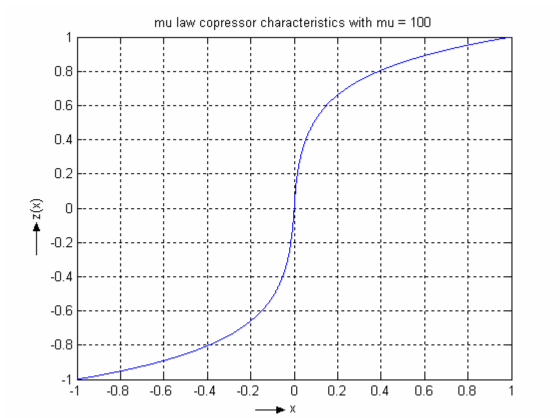

Fig. 3.12.4 *μ-law companding characteristics(mu = 100)*

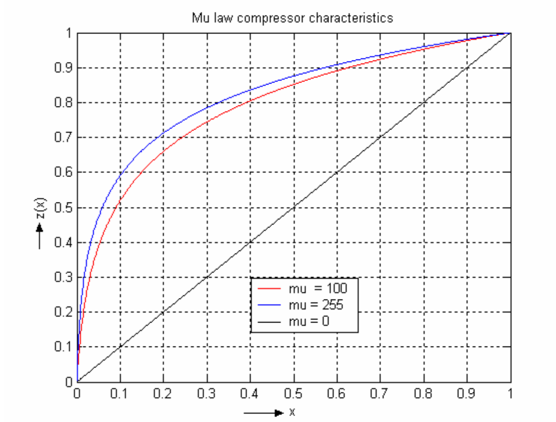

Fig. 3.12.5 *μ-law companding characteristics (mu = 0, 100, 255)*

The compression function c (x) for A - law companding is (**Fig. 3.12.6**):

$$\frac{c(|x|)}{V} = \frac{A\frac{|x|}{V}}{1+\ln A}, \qquad 0 \le \frac{|x|}{V} \le \frac{1}{A}$$

$$= \frac{1+\ln\left(A\frac{|x|}{V}\right)}{1+\ln A}, \qquad \frac{1}{A} \le \frac{|x|}{V} \le 1.0 \qquad 3.12.14$$

'A' is a constant here and the typical value used in practical systems is 87.5.

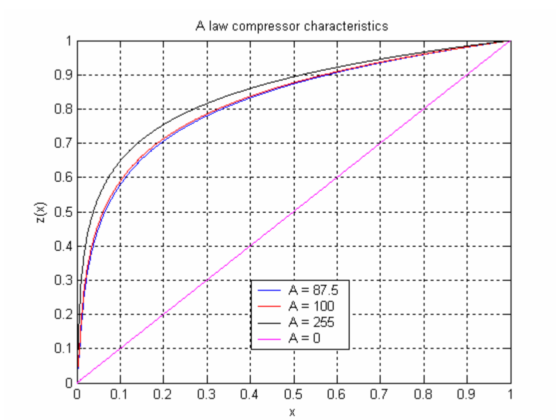

Fig. 3.12.6 *A-law companding characteristics (A = 0, 87.5, 100, 255)*

For telephone grade speech signal with 8-bits per sample and 8-Kilo samples per second, a typical SQNR of 38.4 dB is achieved in practice.

As approximately logarithmic compression function is used for linear quantization, a PCM scheme with non-uniform quantization scheme is also referred as "Log PCM" or "Logarithmic PCM" scheme.

Problems

Q3.12.1) Consider Eq. 3.12.13 and sketch the compression of c (x) for $\mu = 50$ and V = 2.0V

Q3.12.2) Sketch the compression function c (x) for A - law companding (Eq.3.12.14) when V = 1V and A = 50.

Q3.12.3) Comment on the effectiveness of a non-linear quantizer when the peak amplitude of a signal is known to be considerably smaller than the maximum permissible voltage V.