FIITJEE Solutions to IITJEE–2005 Mains Paper

Mathematics

Time: 2 hours

Note: Question number 1 to 8 carries ***2 marks*** each, 9 to 16 carries ***4 marks*** each and 17 to 18 carries ***6 marks*** each.

Q1. A person goes to office either by car, scooter, bus or train probability of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$ respectively. Probability that he reaches office late, if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$ respectively. Given that he reached office in time, then what is the probability that he travelled by a car.

Sol. Let C, S, B, T be the events of the person going by car, scooter, bus or train respectively.

Given that $P(C) = \frac{1}{7}$, $P(S) = \frac{3}{7}$, $P(B) = \frac{2}{7}$, $P(T) = \frac{1}{7}$

Let $\bar{L}$ be the event of the person reaching the office in time.

$\Rightarrow P\left(\frac{\bar{L}}{C}\right) = \frac{7}{9}$, $P\left(\frac{\bar{L}}{S}\right) = \frac{8}{9}$, $P\left(\frac{\bar{L}}{B}\right) = \frac{5}{9}$, $P\left(\frac{\bar{L}}{T}\right) = \frac{8}{9}$

$$\Rightarrow P\left(\frac{C}{\bar{L}}\right) = \frac{P\left(\frac{\bar{L}}{C}\right).P(C)}{P(\bar{L})} = \frac{\frac{1}{7}\times\frac{7}{9}}{\frac{1}{7}\times\frac{7}{9}+\frac{3}{7}\times\frac{8}{9}+\frac{2}{7}\times\frac{5}{9}+\frac{8}{9}\times\frac{1}{7}} = \frac{1}{7}.$$

Q2. Find the range of values of t for which $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Sol. Let y = 2 sin t

so, $y = \frac{1-2x+5x^2}{3x^2-2x-1}$

$\Rightarrow (3y-5)x^2 - 2x(y-1) - (y+1) = 0$

since $x \in R - \left\{1, -\frac{1}{3}\right\}$, so $D \ge 0$

$\Rightarrow y^2 - y - 1 \ge 0$

or $y \ge \frac{1+\sqrt{5}}{2}$ and $y \le \frac{1-\sqrt{5}}{2}$

or $\sin t \ge \frac{1+\sqrt{5}}{4}$ and $\sin t \le \frac{1-\sqrt{5}}{4}$

Hence range of t is $\left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$.

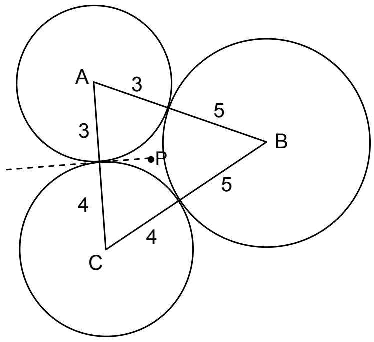
Q3. Circles with radii 3, 4 and 5 touch each other externally if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the points of contact.

Sol. Let A, B, C be the centre of the three circles.
Clearly the point P is the in–centre of the ΔABC, and hence

$$r = \frac{\Delta}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Now $2s = 7 + 8 + 9 = 24 \Rightarrow s = 12$.

Hence $r = \sqrt{\frac{5.4.3}{12}} = \sqrt{5}$.



Q4. Find the equation of the plane containing the line $2x - y + z - 3 = 0$, $3x + y + z = 5$ and at a distance of $\frac{1}{\sqrt{6}}$ from the point $(2, 1, -1)$.

Sol. Let the equation of plane be $(3\lambda + 2)x + (\lambda - 1)y + (\lambda + 1)z - 5\lambda - 3 = 0$

$$\Rightarrow \left|\frac{6\lambda + 4 + \lambda - 1 - \lambda - 1 - 5\lambda - 3}{\sqrt{(3\lambda+2)^2 + (\lambda-1)^2 + (\lambda+1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$\Rightarrow 6(\lambda - 1)^2 = 11\lambda^2 + 12\lambda + 6 \Rightarrow \lambda = 0, -\frac{24}{5}$.

$\Rightarrow$ The planes are $2x - y + z - 3 = 0$ and $62x + 29y + 19z - 105 = 0$.

Q5. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1, 2).

Sol. $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$

$$\Rightarrow \lim_{x_1 \to x_2}\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| < \lim_{x_1 \to x_2} |x_1 - x_2| \Rightarrow |f'(x)| < \delta \Rightarrow f'(x) = 0.$$

Hence f (x) is a constant function and P (1, 2) lies on the curve.
$\Rightarrow$ f (x) = 2 is the curve.
Hence the equation of tangent is $y - 2 = 0$.

Q6. If total number of runs scored in n matches is $\left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$ where $n > 1$, and the runs scored in the k^{th} match are given by $k.\,2^{n+1-k}$, where $1 \le k \le n$. Find n.

Sol. Let $S_n = \sum_{k=1}^{n} k.2^{n+1-k} = 2^{n+1}\sum_{k=1}^{n} k.2^{-k} = 2^{n+1}.2\left[1 - \frac{1}{2^n} - \frac{n}{2^{n+1}}\right]$ (sum of the A.G.P.)

$= 2[2^{n+1} - 2 - n]$

$\Rightarrow \frac{n+1}{4} = 2 \Rightarrow n = 7$.

Q7. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through P (h, k) with the lines y = x and x + y = 2 is $4h^2$. Find the locus of the point P.

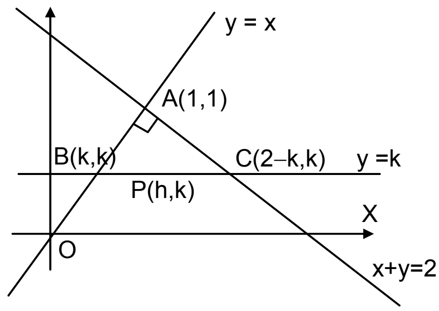
Sol. Area of triangle $= \frac{1}{2}$. AB. AC $= 4h^2$

and AB $= \sqrt{2}\ |k - 1| =$ AC

$\Rightarrow 4h^2 = \frac{1}{2}.\ 2.\ (k - 1)^2$

$\Rightarrow k - 1 = \pm 2h$.

$\Rightarrow$ locus is $y = 2x + 1$, $y = -2x + 1$.



Q8. Evaluate $\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$.

Sol. $I = \int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$

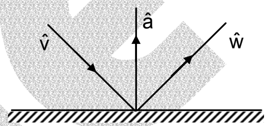
$= 6\int_0^{\pi/2} e^{\cos x}\sin x\cos\left(\frac{1}{2}\cos x\right)dx \qquad \left(\because \int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^{a} f(x)\,dx, & \text{if } f(2a-x) = f(x)\end{cases}\right)$

Let cos x = t

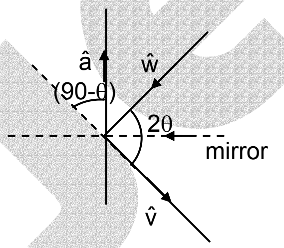
$I = 6\int_0^{1} e^{t}\cos\left(\frac{t}{2}\right)dt$

$= \frac{24}{5}\left(e\cos\left(\frac{1}{2}\right)+\frac{e}{2}\sin\left(\frac{1}{2}\right)-1\right)$.

Q9. Incident ray is along the unit vector $\hat{v}$ and the reflected ray is along the unit vector $\hat{w}$. The normal is along unit vector $\hat{a}$ outwards. Express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.



Sol. $\hat{v}$ is unit vector along the incident ray and $\hat{w}$ is the unit vector along the reflected ray. Hence $\hat{a}$ is a unit vector along the external bisector of $\hat{v}$ and $\hat{w}$. Hence



$\hat{w} - \hat{v} = \lambda\hat{a}$

$\Rightarrow 1 + 1 - \hat{w}\cdot\hat{v} = \lambda^2$

or $2 - 2\cos 2\theta = \lambda^2$

or $\lambda = 2\sin\theta$

where 2θ is the angle between $\hat{v}$ and $\hat{w}$.

Hence $\hat{w} - \hat{v} = 2\sin\theta\,\hat{a} = 2\cos(90^0 - \theta)\hat{a} = -(2\hat{a}\cdot\hat{v})\hat{a}$

$\Rightarrow \hat{w} = \hat{v} - 2(\hat{a}\cdot\hat{v})\hat{a}$.

Q10. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid–point of the chord of contact.

Sol. Any point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ is $(3\sec\theta, 2\tan\theta)$.

Chord of contact of the circle $x^2 + y^2 = 9$ with respect to the point $(3\sec\theta, 2\tan\theta)$ is

$3\sec\theta . x + 2\tan\theta . y = 9$ ….(1)

Let (x_1, y_1) be the mid–point of the chord of contact.

$\Rightarrow$ equation of chord in mid–point form is $xx_1 + yy_1 = x_1^2 + y_1^2$ ….(2)

Since (1) and (2) represent the same line,

$$\frac{3\sec\theta}{x_1} = \frac{2\tan\theta}{y_1} = \frac{9}{x_1^2 + y_1^2}$$

$\Rightarrow \sec\theta = \frac{9x_1}{3\left(x_1^2 + y_1^2\right)}, \ \tan\theta = \frac{9y_1}{2\left(x_1^2 + y_1^2\right)}$

Hence $\frac{81x_1^2}{9\left(x_1^2 + y_1^2\right)^2} - \frac{81y_1^2}{4\left(x_1^2 + y_1^2\right)^2} = 1$

$\Rightarrow$ the required locus is $\frac{x^2}{9}-\frac{y^2}{4} = \left(\frac{x^2 + y^2}{9}\right)^2$.

Q11. Find the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Also find the length of the intercept of the tangent between the coordinate axes.

Sol. Let the equations of tangents to the given circle and the ellipse respectively be

$y = mx + 4\sqrt{1+m^2}$

and $y = mx + \sqrt{25m^2+4}$

Since both of these represent the same common tangent,

$4\sqrt{1+m^2} = \sqrt{25m^2+4}$

$\Rightarrow 16(1 + m^2) = 25m^2 + 4$

$\Rightarrow m = \pm\frac{2}{\sqrt{3}}$

The tangent is at a point in the first quadrant $\Rightarrow m < 0$.

$\Rightarrow m = -\frac{2}{\sqrt{3}}$, so that the equation of the common tangent is

$y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$.

It meets the coordinate axes at $A\left(2\sqrt{7},\ 0\right)$ and $B\left(0,\ 4\sqrt{\frac{7}{3}}\right)$

$\Rightarrow AB = \frac{14}{\sqrt{3}}$.

Q12. If length of tangent at any point on the curve y = f(x) intercepted between the point and the x–axis is of length 1. Find the equation of the curve.

Sol. Length of tangent = $\left|y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right| \Rightarrow 1 = y^2\left[1+\left(\frac{dx}{dy}\right)^2\right]$

$\Rightarrow \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}} \Rightarrow \int\frac{\sqrt{1-y^2}}{y}dy = \pm x + c$.

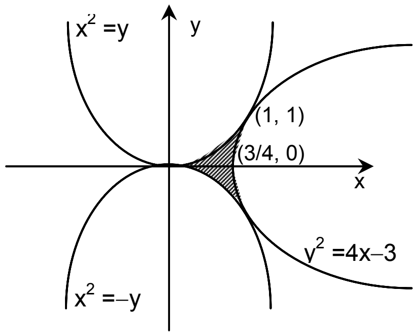
Writing $y = \sin\theta$, $dy = \cos\theta\, d\theta$ and integrating, we get the equation of the curve as

$\sqrt{1-y^2} + \ln\left|\frac{1-\sqrt{1-y^2}}{y}\right| = \pm x + c$.

Q13. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

Sol. The region bounded by the given curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$ is symmetrical about the x–axis. The parabolas $x^2 = y$ and $y^2 = 4x - 3$ touch at the point (1, 1).

Moreover the vertex of the curve $y^2 = 4x - 3$ is at $\left(\frac{3}{4}, 0\right)$.



Hence the area of the region

$= 2\left[\int_0^1 x^2dx - \int_{3/4}^1 \sqrt{4x-3}\,dx\right]$

$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \frac{1}{6}\left((4x-3)^{3/2}\right)_{3/4}^1\right] = 2\left[\frac{1}{3}-\frac{1}{6}\right] = \frac{1}{3}$. sq. units.

Q14. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}\,i$. Find the other vertices of square.

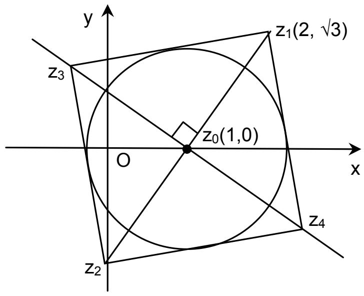
Sol. Since centre of circle i.e. (1, 0) is also the mid–point of diagonals of square

$\Rightarrow \frac{z_1 + z_2}{2} = z_0 \Rightarrow z_2 = -\sqrt{3}i$

and $\frac{z_3 - 1}{z_1 - 1} = e^{\pm i\pi/2}$

$\Rightarrow$ other vertices are

z_3, z_4 = $(1-\sqrt{3}) + i$ and $(1+\sqrt{3}) - i$.



Q15. If f (x – y) = f (x). g (y) – f (y). g (x) and g (x – y) = g (x). g (y) + f (x). f (y) for all x, y ∈ R. If right hand derivative at x = 0 exists for f (x). Find derivative of g (x) at x = 0.

Sol. f(x – y) = f (x) g (y) – f (y) g (x) … (1)

Put x = y in (1), we get

f (0) = 0

put y = 0 in (1), we get

g (0) = 1.

Now, $f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{f(0)g(-h) - g(0)f(-h) - f(0)}{h}$

$= \lim_{h\to 0^+} \frac{f(-h)}{-h}$ $\quad (\because f(0) = 0)$

$= \lim_{h\to 0^+} \frac{f(0-h) - f(0)}{-h}$

$= f'(0^-)$.

Hence f (x) is differentiable at x = 0.

Put y = x in g (x – y) = g (x). g (y) + f (x). f (y).

Also $f^2(x) + g^2(x) = 1$

$\Rightarrow g^2(x) = 1 - f^2(x)$

$\Rightarrow 2g'(0)\, g(0) = -2f(0)\, f'(0) = 0 \Rightarrow g'(0) = 0$.

Q16. If p(x) be a polynomial of degree 3 satisfying p(–1) = 10, p(1) = –6 and p(x) has maximum at x = –1 and p′(x) has minima at x = 1. Find the distance between the local maximum and local minimum of the curve.

Sol. Let the polynomial be $P(x) = ax^3 + bx^2 + cx + d$

According to given conditions

P (–1) = –a + b – c + d = 10

P (1) = a + b + c + d = –6

Also P′ (–1) = 3a – 2b + c = 0

and P″ (1) = 6a + 2b = 0 ⇒ 3a + b = 0

Solving for a, b, c, d we get

$P(x) = x^3 - 3x^2 - 9x + 5$

$\Rightarrow P'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$

⇒ x = –1 is the point of maximum and x = 3 is the point of minimum.

Hence distance between (–1, 10) and (3, –22) is $4\sqrt{65}$ units.

Q17. f(x) is a differentiable function and g (x) is a double differentiable function such that |f (x)| ≤ 1 and f′(x) = g (x). If $f^2(0) + g^2(0) = 9$. Prove that there exists some c ∈ (– 3, 3) such that g (c). g″(c) < 0.

Sol. Let us suppose that both g (x) and g″ (x) are positive for all x ∈ (–3, 3).

Since $f^2(0) + g^2(0) = 9$ and $-1 \le f(x) \le 1$, $2\sqrt{2} \le g(0) \le 3$.

From f′ (x) = g (x), we get

$f(x) = \int_{-3}^{x} g(x)dx + f(-3)$.

Moreover, g″ (x) is assumed to be positive
⇒ the curve y = g (x) is open upwards.

If g (x) is decreasing, then for some value of x $\int_{-3}^{x} g(x)dx$ > area of the rectangle (0 – (–3))2$\sqrt{2}$

⇒ f (x) > 2$\sqrt{2}$ × 3 – 1 i.e. f (x) > 1 which is a contradiction.

If g (x) is increasing, for some value of x $\int_{-3}^{x} g(x)dx$ > area of the rectangle (3 – 0))2$\sqrt{2}$

⇒ f (x) > 2$\sqrt{2}$ × 3 – 1 i.e. f (x) > 1 which is a contradiction.

If g(x) is minimum at x = 0, then $\int_{-3}^{x} g(x)dx$ > area of the rectangle (3 – 0)2$\sqrt{2}$

⇒ f (x) > 2$\sqrt{2}$ × 6 – 1 i.e. f (x) > 1 which is a contradiction.
Hence g (x) and g″ (x) cannot be both positive throughout the interval (–3, 3).
Similarly we can prove that g(x) and g″(x) cannot be both negative throughout the interval (–3, 3).
Hence there is atleast one value of c ∈ (–3, 3) where g (x) and g″ (x) are of opposite sign
⇒ g (c) . g″ (c) < 0.

Alternate:

$$\int_0^3 g(x)dx = \int_0^3 f'(x)dx = f(3) - f(0)$$

$$\Rightarrow \left|\int_0^3 g(x)dx\right| < 2 \qquad \text{……(1)}$$

In the same way $\left|\int_{-3}^0 g(x)dx\right| < 2$ ……(2)

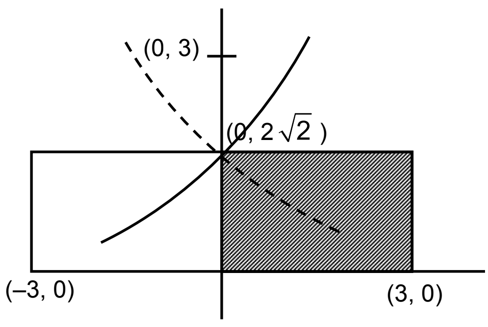
$$\Rightarrow \left|\int_0^3 g(x)dx\right| + \left|\int_{-3}^0 g(x)dx\right| < 4 \qquad \text{……(3)}$$

From $(f(0))^2 + (g(0))^2 = 9$
we get
2$\sqrt{2}$ < g (0) < 3 ……(4)
or –3 < g (0) < –2$\sqrt{2}$ ……(5)

Case I: 2$\sqrt{2}$ < g (0) < 3
Let g (x) is concave upward ∀ x (–3, 3) then the area

$$\left|\int_{-3}^0 g(x)dx\right| + \left|\int_0^3 g(x)dx\right| > 6\sqrt{2}$$

which is a contradiction from equation (3).
∴ g (x) will be concave downward for some c ∈ (–3, 3) i.e. g″ (c) < 0 ……(6)
also at that point c
g (c) will be greater than 2$\sqrt{2}$
⇒ g (c) > 0 ……(7)
From equation (6) and (7)
g (c) . g″ (c) < 0 for some c ∈ (–3, 3).



Case II: $-3 < g(0) < -2\sqrt{2}$

Let g (x) is concave downward $\forall$ x (–3, 3) then the area

$$\left|\int_{-3}^{0} g(x)dx\right| + \left|\int_{0}^{3} g(x)dx\right| > 6\sqrt{2}$$

which is a contradiction from equation (3).

$\therefore$ g (x) will be concave upward for some $c \in (-3, 3)$ i.e. $g''(c) > 0$ ……(8)

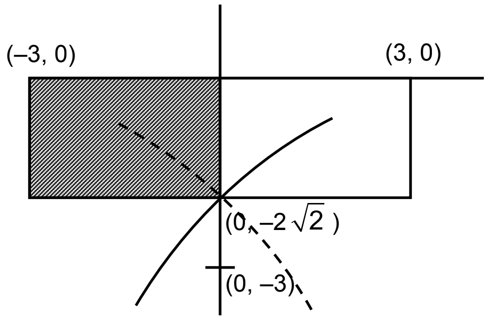
also at that point c

g (c) will be less than $-2\sqrt{2}$

$\Rightarrow g(c) < 0$ ……(9)

From equation (8) and (9)

$g(c) \cdot g''(c) < 0$ for some $c \in (-3, 3)$.



Q18. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f (x) with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f (x) and chord AB.

Sol. Let we have

$4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a$ … (1)

$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b$ … (2)

$4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$ … (3)

Consider a quadratic equation

$4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$

or $[4f(-1) - 3]x^2 + [4f(1) - 3]x + f(2) = 0$ … (4)

As equation (4) has three roots i.e. x = a, b, c. It is an identity.

$\Rightarrow f(-1) = \frac{3}{4}$, $f(1) = \frac{3}{4}$ and $f(2) = 0$

$\Rightarrow f(x) = \frac{(4 - x^2)}{4}$ … (5)

Let point A be (–2, 0) and B be $(2t, -t^2 + 1)$

Now as AB subtends a right angle at the vertex V (0, 1)

$\frac{1}{2} \times \frac{-t^2}{2t} = -1 \Rightarrow t = 4$

$\Rightarrow B \equiv (8, -15)$

$\therefore$ Area $= \int_{-2}^{8}\left(\frac{4 - x^2}{4} + \frac{3x + 6}{2}\right)dx = \frac{125}{3}$ sq. units.