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Entrance Examination, 2005

M.Sc. (Mathematics/Applied Mathematics)

Hall Ticket No.							

Time: **2** hours

Max. Marks: **100**
Part A: **25**
Part B: **75**

Instructions

1. Calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries **1 mark** and each wrong answer carries $-\frac{1}{4}$ **mark**. So do not gamble. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 75 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Do not detach any pages from this answer book. It contains 16 pages. Pages 15 and 16 are for rough work.

Answer Part A by **circling** the correct letter in the array below:

1	a	b	c	d	e
2	a	b	c	d	e
3	a	b	c	d	e
4	a	b	c	d	e
5	a	b	c	d	e
6	a	b	c	d	e
7	a	b	c	d	e
8	a	b	c	d	e
9	a	b	c	d	e
10	a	b	c	d	e
11	a	b	c	d	e
12	a	b	c	d	e
13	a	b	c	d	e
14	a	b	c	d	e
15	a	b	c	d	e
16	a	b	c	d	e
17	a	b	c	d	e
18	a	b	c	d	e
19	a	b	c	d	e
20	a	b	c	d	e
21	a	b	c	d	e
22	a	b	c	d	e
23	a	b	c	d	e
24	a	b	c	d	e
25	a	b	c	d	e

PART A

Find the correct answer and mark it on the answer sheet on the **top page**. A correct answer gets **1 mark** and a wrong answer gets a $-(1/4)$ **mark**.

1. If $\lambda = 1$ is an eigenvalue of the following matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The other two eigenvalues are
(a) 0 and 1.
(b) -1 and 1.
(c) 1 and 2.
(d) -1 and 2.
(e) none of the above.

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}$ is a linear transformation such that the kernel of T is $\{(x, -x) : x \in \mathbb{R}\}$. If T takes $(1, 0)$ to 1, then T takes $(1, 1)$ to which number?
(a) 0. (b) 1. (c) 2. (d) -1. (e) none of these.

3. If there is no solution to the system of equations

$$x - y + kz = 10$$

$$x + y + z = 0$$

$$2x - 2y + z = 15,$$

then the value of k is
(a) 0. (b) $\frac{1}{2}$. (c) -1. (d) 1. (e) none of these.

4. Find all real numbers x such that the matrix

$$\begin{pmatrix} 1 & x & x^2 \\ 1 & 2x & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

is not invertible.
(a) $0, 1, -1$.
(b) $1, 2, 0$.
(c) $0, 1, -2$.
(d) $2, 1, -2$.
(e) none of the above.

5. If $f(x) = \begin{cases} x^b \cos \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$

   then $f'(0)$ exists

   (a) for $b > 0$.
   (b) for $b \geq 1$.
   (c) for no value of b.
   (d) for $b < 0$.
   (e) none of the above.

6. The numbers of complex solutions of the equation $\mathbb{Z}^{10} = (\mathbb{Z} + 1)^{10}$ is

   (a) 0. (b) 10. (c) 9. (d) 11. (e) none of these.

7. Which of the following sets satisfy the condition that for every positive integer n there is some a in A such that $a < n$?

   (a) $A = \{-1, 5\}$.
   (b) A = empty set.
   (c) A = set of all positive integers.
   (d) $A = \{x \in \mathbb{R} : x > 10\}$.
   (e) none of the above.

8. For each positive integer n, let a_n be the number of points of intersection of the graph $y = \sin x$ with the line $y = \dfrac{x}{n}$. The sequence (a_n) is

   (a) decreasing.
   (b) constant.
   (c) converging to zero.
   (d) diverging to infinity.
   (e) none of the above.

9. Let $\sum a_n$ converge and $\sum | a_n |$ diverge. Let $A = \{n \in \mathbb{N} : a_n > 1\}$ and $B = \{n \in \mathbb{N} : a_n < 0\}$. Then it follows that

   (a) both A and B are finite.

   (b) both A and B are nonempty.

   (c) both A and B are infinite.

   (d) A is finite and B is infinite.

   (e) A is infinite and B is finite.

10. The matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

   (a) is not invertible

   (b) satisfies $A^3 = I$

   (c) satisfies $A^2 + I = 0$.

   (d) has real eigenvalues.

   (e) none of the above.

11. The function $e^{x^2} - (e^x)^2$ has differential

   (a) identically zero.

   (b) $2e^{x^2} - 2(e^x)^2$.

   (c) $2x(e^{x^2} - (e^x)^2)$.

   (d) $2xe^{x^2} - 2(e^x)^2$.

   (e) none of the above.

12. Let $f(x) = x + \cos x$ for $x \in \mathbb{R}$. Then f is

   (a) monotonically increasing.

   (b) monotonically decreasing.

   (c) convex.

   (d) concave.

   (e) none of the above.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} -x, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

Then

(a) f is continuous on $\mathbb{R}$.

(b) f is not continuous anywhere in $\mathbb{R}$.

(c) $f(f(x))$ is differentiable in $\mathbb{R}$.

(d) $f(f(x))$ is differentiable only at $x = 0$.

(e) none of the above.

14. The matrix

$$\begin{pmatrix} -1 & 1 & 1 \\ -3 & 4 & 0 \\ -4 & 5 & 1 \end{pmatrix}$$

has rank

(a) 3. (b) 2. (c) 1. (d) 0. (e) none of these.

15.

$$\int_{0.5}^{3.5} (x - [x])dx$$

is equal to

(a) 1.25. (b) 1.06. (c) 1.55. (d) 1.05. (e) none of these.

16. $\lim_{x \to 0} \frac{\sin x^n}{x^n}$ $(n > 0)$

(a) does not exist for all n.

(b) exists only for $n = 1$.

(c) exists and is equal to 0.

(d) exists and is equal to 1.

(e) none of the above.

17. The value of $\int_0^{1/2} \frac{x^2+1}{x^4 - 2x^2 + 1} dx$ is equal to

(a) $\frac{3}{2}$. (b) $\frac{3}{4}$. (c) 1. (d) $\frac{1}{3}$. (e) $\frac{2}{3}$.

18. Let G_1 be a group of order $10^2 + 1$ and G_2 be a group of order $10^2 - 1$. Then

    (a) there is an onto homomorphism from G_1 to G_2.
    (b) there is an onto homomorphism from G_2 to G_1.
    (c) there is a one-one homomorphism from G_2 to G_1.
    (d) there is no onto homomorphism from G_1 to G_2.
    (e) none of the above.

19. The number of non-zero ideals of $\mathbb{Z}/100\mathbb{Z}$ is
    (a) 4. (b) 8. (c) 10. (d) 6. (e) none of these.

20. If G is a cyclic group of order 48 generated by a, and H is the subgroup generated by a^8, then the order of the coset a^2H in G/H is

    (a) 2. (b) 3. (c) 4. (d) 6. (e) 8.

21. I. A subgroup of a cyclic group is always cyclic.
    II. The order of an element in a finite group must divide the order of the group.

    Then

    (a) both I and II are true.
    (b) I is true but II is false.
    (c) I is false but II is true.
    (d) both I and II are false.
    (e) I may sometimes be true but II is always false.

22. In the set $\mathbb{Z}_{15}$ of integers modulo 15, the number of elements x such that $x^2 \equiv 1 (\text{mod} 15)$ is

    (a) 2. (b) 3. (c) 4. (d) 5. (e) none of these.

23. Two subsets A and B of $\{1, 2, 3, 4, 5, 6\}$ are chosen at random. What is the probability that $A \cap B = \{6\}$ and $A \cup B = \{4, 5, 6\}$.

    (a) $\frac{1}{16 \times 64}$. (b) $\frac{1}{16 \times 9}$. (c) $\frac{1}{16}$. (d) $\frac{1}{16 \times 32}$.

    (e)none of the above.

24. If $P(A) = 0.6$, $P(B) = 0.8$, then about $P(A^c \cup B^c)$ we can say that

(a) it is at least 0.6.
(b) it is at most 0.6.
(c) it is exactly equal to 0.4.
(d) it is exactly equal to 0.6.
(e) nothing can be said from the information given.

25. X is a random variable with the following probability distribution. $P(X = -\frac{1}{2}) = \frac{1}{3}$, $P(X = -1) = P(X = 1) = \frac{1}{6}$, $P(X = 2) = \frac{1}{3}$. Then

(a) X and $-X$ are identically distributed.
(b) $\frac{1}{X}$ and X are identically distributed
(c) $-X$ and $\frac{1}{X}$ are identically distributed.
(d) $\frac{1}{X}$ and $-\frac{1}{X}$ are identically distributed.
(e) none of the above.

PART B (pages 7 to 14)

There are 15 questions in this part. Each question carries 5 marks. Answer as many as you can. The maximum you can score is 75 marks. Brief proofs are needed for each question in the place provided below the question.

1. Let $f(x)$ and $g(x)$ be two polynomials with complex co-efficients. If $x^2 + x + 1$ divides $f(x^3) + x^2 g(x^3)$, then show that $f(1) = g(1) = 0$.

2. Let $f(x) = \dfrac{x}{1+|x|}$ for all x in $\mathbb{R}$. Answer the following questions.
   (i) What is the range of f?
   (ii) Is f one-one?
   (iii) Is f increasing?
   (iv) Find x for which $f(f(x)) = -\dfrac{1}{3}$.
   (v) Find all points x such that $f(x) = x$.

3. Let $f : [0,1] \to [0,1]$ be a continuous onto function. If $f(0) = f(1) = 0$, then show that there exist two distinct numbers $x, y \in [0,1]$ such that $f(x) = f(y) = \frac{1}{2}$.

4. Let (a_n) be a sequence of numbers from $[0,1]$. Suppose $a_{n+1} = \sqrt{2a_n - 1}$ for $n = 1, 2, \ldots, \infty$. Examine whether (a_n) is convergent. If it is convergent, find $\lim_{n\to\infty} a_n$.

5. Find a polynomial function f from $\mathbb{R}$ to $\mathbb{R}$ with the following properties

$$f(0) = 1 = f(2) \text{ and } f'(0) = 1.$$

6. Let G be a finite group of odd order. Show that the function $\phi : G \to G$ given by $\phi(g) = g^2$ is one-one and onto.

7. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ be a polynomial with integer co-efficients. If $f(1)$ is odd and $f(0) = 3$, then show that $f(X)$ cannot have an integer root.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be any injective map such that

$$f(xy) = f(x)f(y) \quad \text{and} \quad f(x+y) = f(x) + f(y).$$

Show that if $x < y$ then $f(x) < f(y)$.

9. Let A and B be two events with $P(A) = P(B) = 1$. Compute $P(A \cap B)$. Examine whether the two events A and B are independent.

10. Given that $X \sim B(n, \frac{1}{2})$, evaluate the correlation coefficient between $Y_1 = X + 1$ and $Y_2 = \frac{1}{X+1}$.

11. Let $y = f(x), 0 < x < 3$ and suppose you know there is only one value $x_0, 0 < x_0 < 3, x_0 \neq 1, 2$ such that $f'(x_0) = 0$ and that $f(x_0)$ is a minimum value. Suppose the graph of $y = f(x)$ passes through $(1, 1)$ and $(2, 2)$. Show that x_0 cannot be greater than 2.

12. Give an example of a finite group that is not abelian. Give two elements $a, b \in G$ such that $ab \neq ba$. What are the orders of ab and ba in the example that you give? Do ab and ba always have the same order? Why?

13. Consider the ring $\mathbb{Z}_{20}$ of integers modulo 20. List all the zero divisors in $\mathbb{Z}_{20}$. Show that if a is not a zero divisor, then it has a multiplicative inverse.

14. Obtain the equation of the plane which passes through the point $(-1, 3, 2)$ and is perpendicular to each of the two planes $x+2y+2z = 5$ and $3x + 3y + 2z = 8$.

15. Consider two linear transformations T_1 and T_2 from $\mathbb{R}^3$ to $\mathbb{R}^3$

$$T_1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + \sqrt{2}x_2 + 2x_3 \\ \sqrt{2}x_1 + 3x_2 + \sqrt{2}x_3 \\ 2x_1 + \sqrt{2}x_2 + x_3 \end{pmatrix},$$

$$T_2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + 2x_2 \\ 2x_1 + 3x_2 \\ -x_3 \end{pmatrix}.$$

Are these two linear transformations similar?

ROUGH WORK

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