Faculty of Actuaries **Institute of Actuaries**

ADVANCED CERTIFICATE IN DERIVATIVES: FURTHER MATHEMATICS, PRINCIPLES AND PRACTICE

Specimen Examination Paper

April 1999

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *You have 15 minutes at the start of the examination in which to read the questions. You are strongly encouraged to use this time for reading only but notes may be made.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*

4. *Mark allocations are shown in brackets.*

5. *Attempt all 6 questions leaving sufficient space between each answer.*

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available actuarial tables and an electronic calculator.

1 You have been asked to assess the adequacy of policies, procedures and management information systems for credit risk management at a bank that actively markets derivative products to customers.

(i) Draw up a list of items that you would request from management as part of your assessment of the bank's credit risk management. [3]

(ii) Describe the investigations that you would make as part of this assessment. [14]

[Total 17]

2 Consider a German government bond yield curve, let $P(t)$ be the price and $y(t)$ the yield of a zero coupon bond of length t, where $t = 1, 2, 3$ ... years. Assume intermediate points along the curve are not required.

(i) Derive formulae for $f(t)$, the one-year forward rate from $t - 1$ to t, and $g(t)$, the par coupon yield at time t, in terms of $P(t)$ for $t = 1, 2, 3 \ldots$. The par coupon yield is the annual coupon on a bond priced at par (DM100 per DM100 nominal). [3]

(ii) (a) Calculate the values of the forward rates and par yields given the following annual zero coupon German government bond yields:

Maturity (years)	*Zero coupon yield*
1	4.20%
2	4.70%
3	4.90%
4	4.95%
5	4.90%

(b) Sketch the zero coupon, par and forward rate yield curves on a single graph and explain the relationship between the shape of the par and zero coupon yield curves. [11]

(iii) Use your curve to value a DM 100 million 5-year interest rate swap, paying 6% fixed rate annually and receiving floating rate annually, and suggest reasons why in practice the market value of such a swap might be different. [5]

[Total 19]

3 (i) (a) List the problems that arise for a large unit-linked life fund in moving monies between the major equity markets by buying and selling the underlying stocks. Your answer should address both short-term switches which are intended to be reversed and long-term strategic asset allocation switches.

(b) Describe how the problems in (a) can be overcome using equity index futures contracts. [8]

(ii) You are the head of derivatives trading at a UK fund management firm. You are approached by a single-A rated counterparty with which neither you nor any member of the firm are familiar. This counterparty is willing to enter into a one-year swap in which it will pay the fund the total return on a suitable basket of UK gilts in return for the fund paying it the total return on a suitable basket of UK equities, plus (or possibly minus) an adjustment to reflect the "market rate" for such a swap in the market.

Discuss how in theory you would estimate the appropriate size of the adjustment. [6]

[Total 14]

4 A pension fund wishes to use futures to hedge (against interest-rate movements) a £10,000,000 nominal holding of a sterling corporate bond with a dirty price of £107.50, a duration of 4.41 years and a gross redemption yield of 5.84% semi-annual. The cheapest-to-deliver gilt for the current active LIFFE Gilt future has a dirty price of £119.30, a duration of 6.88 years and a gross redemption yield of 4.62% semi-annual. The price of this LIFFE Gilt future is £117.40, and the conversion factor for the cheapest-to-deliver is given as 1.0164.

(i) Calculate the number of Gilt futures that the pension fund should sell in order to hedge the corporate holding. Clearly identify any assumptions you make. [4]

(ii) Describe the possible risks to the pension fund if it proceeds with the hedge in the amounts you calculate for (i). [4]

[Total 8]

5 (i) You have been asked to consider a model for pricing and hedging vanilla and exotic interest-rate sensitive options. Discuss the rationale for using a single factor model of the short-term interest rate which covers the entire yield curve, and state potential features which would be desirable.

It has been proposed to use a model based on the standard form:

$$dr = \mu(r, \sigma, t)dt + \sigma(r, t)dz$$

where r is the short rate, t is time and z is a stochastic random variable following a Wiener process.

Describe, with brief examples, the different ways in which published models of this form have approached the issues of (a) negative interest rates and (b) the wide "dispersion" of rates over time due to future uncertainty. [12]

(ii) The Cox-Ingersoll-Ross (CIR) model is a one-factor interest-rate model of the form:

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz$$

which leads to the solution (at time 0) for the prices $P(t)$ of zero-coupon bonds of maturity t:

$$P(t) = A(t).\exp(-rB(t))$$

where

$$A(t) = \left\{ \frac{\phi_1 \exp(\phi_2 t)}{\phi_2[\exp(\phi_1 t) - 1] + \phi_1} \right\}^{\phi_3} \quad \text{and} \quad B(t) = \frac{\exp(\phi_1 t) - 1}{\phi_2[\exp(\phi_1 t) - 1] + \phi_1}$$

with ϕ_1, ϕ_2 and ϕ_3 all positive constants ($\phi_1 > \phi_2$) based on interest-rate process parameters κ, θ, σ and an imposed "market price of risk" parameter λ.

Describe how this model meets (or fails to meet) the criteria you listed in (i), and prove that, for large t:

$$P(t) \approx \exp(-R\,t)$$

where $R = (\phi_1 - \phi_2)\,\phi_3$ is a constant long-term rate. [8]

(iii) Cox-Ingersoll-Ross is both an "equilibrium" model and a "no-arbitrage" model. Outline briefly what you understand by these terms, and the significance of such properties for an interest-rate model. [4]

[Total 24]

6 You are the risk manager in charge of a fund invested exclusively in UK equities.

(i) Describe the concepts behind VaR (value at risk) as a risk management tool for your fund. [3]

(ii) Discuss, in the form of brief notes, the advantages and disadvantages of using the following methods to assess market price movements in your VaR model:

(a) compiling a complete variance-covariance matrix based on historical movements of each sector of the equity market;

(b) recalculating the profit and loss on the current portfolio using historical prices of the individual equities or equity indices;

(c) running a Monte-Carlo simulation using a time-series model (such as GARCH) to forecast future volatilities. [10]

(iii) The portfolio also contains a number of long-dated vanilla and exotic options on individual equities. Explain briefly how the usual option sensitivities (delta, gamma etc.) might be inappropriate for calculating VaR for this part of the portfolio. Devise a better procedure using scenario analysis and the historical simulation method from (b) above. 5]

[Total 18]

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1 (i) Management should be asked for the following items:

- details of the method of measurement of credit risk
- credit management policies and procedures
- organisation chart
- credit risk reports used by line management (including details of limit monitoring)
- exceptions reports covering credit risk and
- deteriorating trend reports
- examples of documents, including netting agreements

(ii) **To determine if there is proper segregation between the marketing and credit risk management functions, review the structure of the organisation.**

- Is the credit risk management function independent of the trading and marketing function?
- Make sure that the credit risk management group has a significant input into the new product approval procedure.
- Assess the level of knowledge and experience of the credit risk management team.
- The credit risk management team must have sufficient resources, authority and access to senior management.

Assess the adequacy of credit risk management policies and procedures for derivative activities.

- Ensure that credit risk management policies are approved by the board annually.
- Are there guidelines for the credit quality of the derivative portfolio in terms of concentrations, terms and trends?
- Are there periodic reviews of counterparties and subsequent assignment of risk weightings?

- What methodology and source data is used to calculate counterparty credit risk exposure and how often is it validated?

- What reporting requirements exist for counterparty credit exposure?

- Are there separate counterparty limits for pre-settlement and settlement credit risk?

- What procedures are there in place for reporting credit limit exceptions and who has authority to approve such limit exceptions?

- Is there independent monitoring of the aggregate exposure to counterparties across all business lines against limits?

- Are customers' transactions appropriate given their own policies, procedures and risk profiles?

- Examine credit enhancement issues. (E.g. when is collateral required.)

- How are non-performing contracts being handled?

- Look at the general credit allowance (which will be earned over time as a compensation for being exposed to credit risk) and the amount, if any, to cover probable credit losses. These allowances should be reflected in pricing.

Does the credit risk measurement method adequately assess both current and potential exposure?

- Are the calculations performed sufficiently frequently, particularly where a counterparty is close to their credit limit and/or following large market movements?

- Does the method produce estimates of credit exposure appropriate to the capital base of the bank?

- Has the model been reviewed by an independent party?

Carry out a review of credit administration procedures

- How often are counterparty credit reviews carried out? Once a year is the absolute minimum frequency.

- Are the limits and approvals in place for a counterparty before a deal is done?

- Have staff in the credit risk management area demonstrated the ability to control and limit exposures to counterparties whose credit quality has deteriorated?

Analyse derivative counterparties by dealer.

- Ensure that files are up-to-date and that they contain enough information to make an informed credit decision.
- Make sure that a proper assessment of credit risk has been carried out and that the limits and risk ratings are appropriate in the light of this assessment.
- Has the dealer considered the appropriateness of the use of derivatives by the counterparty and its likely impact on the counterparty's financial position?

Look at the reports that senior management get on credit risk.

- Do the reports give a true picture of the bank's exposure to each counterparty and to all counterparties in aggregate?
- Are significant credit exposures shown in the reports?
- Are exposure trends analysed by country, industry, credit rating and maturity?
- Are they produced frequently and quickly enough?

Master netting agreements

- Does the bank document all its transactions under master netting agreements?
- Is there adequate legal opinion to support the enforceability of netting in the various jurisdictions in which it is used?
- Sample counterparties to see if in fact there are signed master netting agreements in place.
- Examine the extent of documentation backlog and the procedures management has in place to track and get outstanding agreements signed up.

Review of operations

- Make sure that the people who perform the credit exposure calculations are both competent to do so and independent of the trading and marketing function.

- Is the bank's credit exposure model able to cope with the required volume of calculations demanded by the business?

- Are credit line limits and the use of credit lines input into the system in a timely fashion?

Contingency plans

- Are there contingency plans to deal with a major credit deterioration or a massive disruption in the market?

Compliance with stated policy

- How has the bank complied with its own stated policy on credit risk management?

- How do traders verify credit line limits and usage?

- How are credit limit excess dealt with in terms of approval and action to bring the credit exposure back within normal limits?

- What would prevent a trader dealing with a counterparty for which there is no credit line?

Have there been any recent credit downgrades?

- How did the bank respond compared with its stated credit risk management policy?

2 (i) $P(t) = (1 + y(t))^{-t}$ for t = 1, 2, 3 etc.

and $P(0) = 1$

$$f(t) = \frac{P(t-1)}{P(t)} - 1$$

$$1 = \sum_{s=1}^{t} g(t)P(s) + 1.P(t)$$

$$\Rightarrow g(t) = \frac{1 - P(t)}{\sum_{s=1}^{t} P(s)}$$

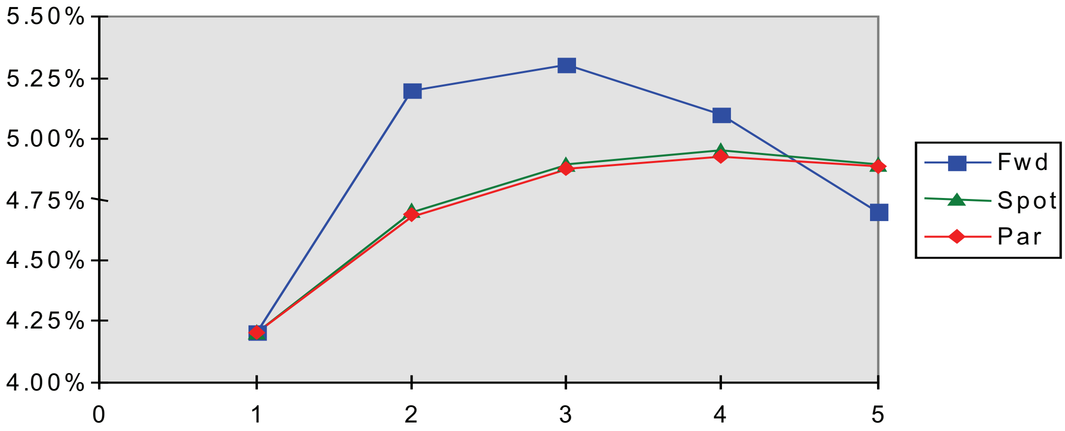
In the above, $P(t)$ are in decimal (i.e. par = 1). The candidate could also obtain full marks using par = 100 and putting the rates in %.

(ii) (a) The following table should be created (the final column is only for section (iii) below):

t	*y(t)*	*P(t)*	*f(t)*	*g(t)*	*Swap flows*
0		1			
1	4.200%	0.95969	4.200%	4.200%	−1.800%
2	4.700%	0.91223	5.202%	4.688%	−0.798%
3	4.900%	0.86631	5.301%	4.882%	−0.699%
4	4.950%	0.82427	5.100%	4.933%	−0.900%
5	4.900%	0.78727	4.700%	4.891%	−1.300%

Note: although not part of the question — that at $t = 1$ the slope of the forward rates $f(t)$ is about twice that of $y(t)$: $5.202 - 4.2 \approx 2(4.7 - 4.2)$. It is quite easy to show that this is true at the starting point of all yield curves.

(b)



The par and spot curves are virtually identical, with the spot slightly above the par.

The similarity is because the yield curve is valuing all cashflows on the same date at the same rate. To first order all the yields are very similar, so any bond of the same maturity will have approximately the same yield.

The reason for the slight difference is that, in an upward sloping yield curve, the par bond has a lower reinvestment rate for its coupon in the early years, whereas the zero has its yield throughout (by definition).

The last two paragraphs are slightly more detailed than required for full marks.

(iii) The swap floating coupons are the $f(t)$.

So the value of the swap is:

$$\text{DM } 100{,}000{,}000 \sum_{s=1}^{5} -(6\% - f(s))P(s)$$

and the values of minus ($6\%–f(s)$) are given in the last column of the table above.

Hence value = DM 100,000,000 × –0.048262 = DM – 4,826,200.

Note that the swap value is negative.

An equivalent method would be to derive the answer as the difference between the value of a floating-rate note (at par) and the value of a 6% coupon bond. Candidates who use a different day count basis would not be penalised.

The true market value will be different because:

- the curve given was a government curve and swaps are traded on an **interbank curve**, which has higher yields (lower credit quality) — this would make the swap value less negative

- **day count convention**: the formulae assume interest rates are based on Actual/365 (or Actual/Actual) day count, but in fact DM swaps are based on Actual/360 for the floating part and (usually) 30/360 or Actual/360 on the fixed part — this alters the payment schedule. Changes in day count conventions have surprisingly large effects on swap values.

3 (i) The main problems in moving monies between markets for both long and short term switches are as follows:

- the costs (commission, bid-offer spread, purchase taxes, etc.) in switching between markets and reversing the switch within a few months can be significant

- short term switches may upset a strategic profile of stocks in a market

- the operational aspects of carrying out a decision can be slow with the consequent loss of some of the benefits of the decision

- the back office may be over stretched leading to a higher risk of errors if several stocks are being bought and sold at once

- lack of liquidity and depth in the underlying markets can reduce flexibility

- the taxation impact of selling shares which have shown significant appreciation in price may be unacceptable in terms of cost

For a short term switch, bid/offer spreads, commissions and any lack of liquidity and depth will cost the manager four-fold in a round trip between two markets.

Using stock index futures the manager can adjust and subsequently re adjust the portfolio’s exposure between the two markets at a significantly lower cost.

No tax is crystallised on equity capital gains and the long term profile of the fund remains in tact.

The investment decision can be executed immediately to catch all the anticipated movements in both markets whereas otherwise some of the benefits of the decision could be lost because of the time taken to process sales in the underlying stocks in the relevant markets.

In the case of a short term switch there would be of the order of four contract notes to be processed for each stock in one of the markets (assuming say 25 stocks are held in each market this would run to 100 contract notes to be processed); using index futures only four contract notes need to be processed.

Futures markets are often more liquid than the market in the underlying stocks so it is possible to deal in size without moving the market. Stock index futures avoid the need to trade the underlying stocks and thereby avoid the movement in the market prices of stock associated with trading large volumes.

In a very large investment house it may be virtually impossible to make substantial asset allocation switches without the use of futures.

For a long term switch between markets, stock index futures can also be very useful.

The switch can be achieved by selling stock index futures in the market the manager wishes to reduce his exposure to (say the US) and buying stock index futures in the market to which the manager wished to increase his exposure (say the UK).

With this strategy, the manager is protected from falls in the US market because losses on the underlying are made up by gains on the short position.

Gains on the UK market accrue to the fund through the long futures position.

Having locked in his strategic asset profile the manager can now comfortably proceed with stock switching and unwind the futures positions appropriately as he proceeds. The fact that the manager does not have to sell large volumes of stock quickly, should allow the manager to do individual stock deals on more advantageous terms.

This strategy allows the manager to ensure he locks in his long term view without losing the market opportunity while trying to fine tune his stock selection and switching process.

As equity transactions are spread over a longer time period the pressure on the back office is reduced.

(ii) The swap in question would often be considered as two separate total return swaps (TRSs), one paying equity total return in exchange for LIBOR, the other receiving gilt return in exchange for LIBOR. This is also a helpful way of viewing the transaction, except there is a reduction in funding cost (see below).

In the absence of distortions caused by tax, transaction costs and credit risk adjustments, a contract which swaps one market return (equity or gilt) for another (LIBOR) would have zero cost. This is because each TRS can be replicated by going long of the first asset and funding the cost at LIBOR, as in fact is normally done as a hedge.

In practice, it is usually necessary to take into account the implied tax position of the average investor with regard dividends. This sort of adjustment is unlikely to be much in this instance but it is a feature of competitive pricing.

Transaction costs represent the cost of setting up and running the hedge to your firm, together with a profit margin.

For the equities, the basket could be replicated exactly by purchasing the individual shares in the relevant proportions and borrowing the purchase cost. There would be a purchase and re-sale bid-offer spread.

For the gilts, the replication would be by selling, with another bid/offer spread, but additionally there would be the cost of borrowing the bonds during the life of the TRS. In the repo market, this is normally effected by a reduction in the funding rate paid on the cash raised.

The balance of the two LIBOR rates, plus the bid/offer spreads, would result in a cost to the counterparty. This margin could be reduced if your firm is already short of equities or long of gilts.

[*In single asset TRSs, the cost can often be reduced if one side is already long of cash, since then LIBOR becomes LIBID. But, as the swaps are done back-to-back, there would be a saving on the LIBOR bid/offer (pay-receive) spread. The total return reset dates (on which the asset total return is assessed and settled and possibly rolled over) will probably be annual - they do not want to be too often, as valuation and transaction costs will be incurred. However, the LIBOR resets would usually be more frequent e.g. quarterly, so that the funding could be locked in for a liquid market period. There is thus a reset date mismatch.*]

We should also consider how in reality the TRSs are to be replicated. Buying and selling shares in bulk is expensive. It might be possible, if the basket of equities closely tracks the FTSE100 index, to use the FTSE future as a hedge. However, there are problems with using futures: basis risk (timing of purchase and sale vs fair value of the future) and tracking risk. The answer to Q4 covers these.

It is also sometimes possible to trade in forward contracts for certain asset baskets. In addition, however, futures and forwards pay no income, so there is a problem of estimating dividends. This is a major risk in equities. It may be that there is synergy with the options desk, whose problem is often the opposite (trying to exclude the value of dividends from their hedge).

Given this risk, you might conclude that it is not worth using futures or forwards to hedge. But if other similar firms do, to be competitive on pricing you might have to. There is also a saving on balance sheet usage if a future is used instead of the cash market.

Allowing for credit risk seldom affects the pricing. This is because the market norm is to use collateral, which adjusts within the TRS reset period for the accrued total return to date by means of a cash balance (if equities fall substantially, the total return receiver might have to pay a net balance to the payer!). Over-collateralisation (haircuts) would be recommended for risky or lesser known counterparties.

However, if no collateral were to be paid then some adjustment to the funding rates would be required, resulting in a higher margin paid by the counterparty. It is not a trivial exercise to assess this accurately for a two asset TRS. In theory, a two-factor model could be used to project the pattern of future credit risks.

[*Note that the major credit risk on a single asset TRS without collateral is that both (a) the total return on the asset is negative (i.e. the market falls or there is a default) and (b) the receiver defaults on the swap. If none or only one happens, there is no major credit loss.*]

Finally, a suitable profit margin should be added to reflect (i) the cost of capital used in the trade and (ii) the degree of mismatch risk undertaken with the hedge.

4 (i) The principle of hedging against interest movements is that, for two portfolios Π_1 and Π_2, the original and hedging portfolios respectively, we need $\Delta\Pi_1 = \Delta\Pi_2$. We can only achieve this using duration analysis if we specify that the underlying interest rate (yield) changes Δy_1 and Δy_2 are proportional. It is usual to assume that $\Delta y_1 = \Delta y_2$, i.e. the yield curve moves in parallel.

Then, since the modified duration $w_i = -\frac{1}{P_i}\frac{dP_i}{dy_i}$ and for amounts X_i of a bond in portfolio i with small changes ΔP_i, $\Delta\Pi_i = X_i \Delta P_i$, we thus have:

$$X_1 P_1 w_1 \Delta y_1 = X_2 P_2 w_2 \Delta y_2$$

and so a good hedge is:

$$X_2 = X_1 \cdot \frac{P_1 w_1}{P_2 w_2}.$$

It is not necessary to include all the above theory in any answer.

Hence for this question, considering the amount of cheapest-to-deliver gilt required to hedge the corporate bond:

$$X_{Gilt} = £10{,}000{,}000 \times \frac{107.50}{119.30} \times \frac{4.41\left(1+\frac{0.0584}{2}\right)}{6.88\left(1+\frac{0.0462}{2}\right)} = £5{,}810{,}318$$

converting (Macaulay) duration from the question to modified duration.

This hedge amount assumes that the two yields curves, corporate and gilt, move in parallel equal amounts during the life of the hedge.

To obtain the number of futures, we can take one of two approaches.

(a) The price of the future could replace the price of the cheapest-to-deliver bond in the above formula. This assumes that the duration of the future is the same as that of the cheapest-to-deliver (i.e. ignores convexity) and gives the solution:

$$X_{Future} = X_{Gilt} \times \frac{119.30}{117.40} = £5{,}904{,}352$$

or 59 contracts (each Gilt contract being £100,000).

(b) The more usual (correct?) method is to adjust the holding by the conversion factor:

$$X_{Future} = X_{Gilt} \times 1.0164 = £5{,}905{,}607$$

or 59 contracts again. Methods (a) and (b) only differ significantly when the cheapest-to-deliver price is very different from the future's price, but (b) is always more precise. Note that method (b) does not in fact use the future's price at all.

Both (a) and (b) assume that there will be no basis movement between futures and cheapest-to-deliver between now and the delivery date of the future.

(ii)

- The hedge only works for small Δy, as w changes with y and we are effectively approximating to a curve with a straight line. For a large shift in yields, the convexity (second-order) effects will become significant.

Some form of dynamic hedging is normally used, i.e. re-adjusting the hedge occasionally. There is a cost to this, though, which can be measured in option theory terms, since the convexity effect acts like a short or long option position.

- The hedge is only one factor (a simple yield curve shift). If the maturity of the two bonds is mismatched, as in the question where durations are very different between the two, there will be a sensitivity to a tilt in the yield curve which is not allowed for in the hedge.

  The hedge takes no account of credit risk. The corporate bond yield will move independently from market movements due to variations in expectations of default risk (known as the yield spread, or credit spread). Interest-rate hedging cannot remove the risk of default.

- Adverse basis risk may arise if the future is sold when cheap or re-purchased when expensive. Basis risk is the movement of futures against "fair value" due to supply and demand effects.

- Futures gains and losses must be settled as a cash margin. If the futures position goes far away from the entry price, a large interim cash balance will be generated without an equivalent counterpart from the bond. This has to be funded. Such a feature sunk the otherwise effective Metallgesellshaft oil price hedge in 1994 (see Dowd p124).

5 *Most of this answer can be found in Chapter 11 of Rebonato. Although the CIR model is not widely used, the points Rebonato raises in the chapter are generally applicable.*

(i) The **rationale** for a full yield curve model is to be able to price simultaneously options which cover a wide spectrum of maturities based on a single currency yield curve. Potential exotic types of options include yield spread options and path-dependent options, such as multiple date knock-in (or -out) options. These depend not just on the movement of rates along the curve, but the correlation between the movements.

A single factor model of the short rate *r* creates a process for *r* which leads to an evolution of short rates over time. Parameters (which can be time- and curve-dependent) govern the evolution. From this evolution, all present and future bonds and swap rates can be priced. The success, or otherwise, of the model can be measured in terms of how well it prices, with one parameter set, the full range of securities and options observable in the market.

Single factor models are usually easy to use and calibrate, and can even in some simple cases lead to analytical solutions. Where numerical techniques are needed, the single factor means that the dimension of the problem is as low as possible, so calculations are faster and the time steps can be smaller than for two or more factors.

Such models are normally calibrated to the current yield curve, so that the model correctly prices discount bonds for the range of maturities required. (See answer to (iii) for more on this point.) There are usually some degrees of freedom available — more time dependence of the parameters could be added — to also calibrate the model to some observed set of contingent claims e.g. caps.

Desirable features are:

- ease of use and calculation, especially calibration - e.g. this might lead to a log-normal model for rates to fit easily to the standard Black pricing for caps

- the current swap (or bond) curve should be reproduced by the model

- imperfect correlation between forward rates (yield spread options can't be priced properly without this) — this is only really needed when pricing certain types of option e.g. caps can be quite happily priced as individual caplets

- no negative interest rates

- reasonable dispersion of rates over time (due to the Brownian motion), otherwise too large a probability of getting an absurdly high or low value — market prices do not allow for these extremes

- another way of expressing the previous point is to say that, when rates go too high (or low), they tend to revert back to some middle level ("mean-reverting")

- volatility of rates of different maturity should be different, with shorter rates usually being more volatile

- volatility of rates is thought to depend on the absolute level of rates, i.e. high rates ⇒ high volatility, but this is not universally agreed.

(a) **Negative rates** can be avoided by making the random (dz) process log-normal, i.e.

$$\frac{dr}{r} = \mu(\sigma, t)dt + \sigma(t)dz$$

or, equivalently,

$$d(\log_e r) = \left\{\mu(\sigma, t) - \tfrac{1}{2}\sigma^2(t)\right\}dt + \sigma(t)dz\,.$$

This is used in the Black-Derman-Toy model.
Other possible forms include versions which have the term $\sigma(r,t) = \sigma(t)r^{\beta}dz$ for some $\beta > 0$ (not necessarily an integer) which is the general form of the Hull-White one-dimensional model. Note that $\beta = 1$ gives a log-normal model.

(b) **Over-wide dispersion** is resolved by one of two methods: (I) constraint on the volatility function by making $\sigma(t)$ a decreasing function of t, and/or (II) making the drift process μ mean-reverting. Mean-reversion introduces a form for the drift parameter μ of:

$$\mu(r,t) = a(t)\big(b(t) - r\big)$$

where b is the (moving) long-term target for r and a is the speed (or force) of the correction. (Even more complicated forms are possible.)

Models which have focused on decaying variance are BDT (one its main weaknesses, in fact) and Ho&Lee, whilst those using an explicit mean-reversion are CIR, Vasicek and one-factor simple and extended Hull-White.

(ii) CIR satisfies most of the desirable features above. It is easy to use (algebraic), has time-dependent volatility, no negative rates and copes with dispersion through mean reversion.

It has no de-correlation of forward rates, and scores less well on the possible shapes of yield curve it can fit precisely (in fact, only simple upward sloping, simple downward sloping and single-humped are possible).

Let $R(t) = -\frac{1}{t}\ln P = \frac{-\ln A - Br}{t}$.

Now, $\ln A = \phi_3\left(\ln(\phi_1) + \phi_2 t - \ln[\phi_2(\exp(\phi_1 t) - 1) + \phi_1]\right)$
$\rightarrow \phi_3\left(\phi_2 - \ln[\phi_2(\exp(\phi_1 t)]\right)$ as $t \rightarrow \infty$, since terms in t dominate the constant terms, so

$$\ln A \rightarrow \phi_3\left(\phi_2 t - \ln\phi_2 - \phi_1 t\right) \rightarrow \phi_3\left(\phi_2 - \phi_1\right)t$$

and, since $B \rightarrow \frac{1}{\phi_2}$ as $t \rightarrow \infty$ and r is a constant at time zero, $\frac{-Br}{t} \rightarrow 0$ as $t \rightarrow \infty$, hence $R(t) = \frac{-\ln A - Br}{t} \rightarrow \phi_3\left(\phi_1 - \phi_2\right)$ as $t \rightarrow \infty$, which is a (positive) constant.

[$\ln \equiv \log_e$]

The notation in this question uses t in place of Rebonato's τ = t – T, so the question is only looking at very large T with t = 0. Hence, for example, r in the above is actually r(0) in Rebonato's text.

(iii) **Equilibrium** models are a particular class of models where a simple form of an entire economy is described by the model. To obtain a parsimonious model (few parameters) the economy is often simplified to consist only of a single production source and a single "good" which can be consumed. Deferred consumption (investing) enables higher future consumption, and is governed by a utility function.

All securities and contingent claims are priced endogenously in this model. This gives a world of "absolute" pricing.

However, in practice, the simplifications of the economy are so great that only certain yield curve shapes are possible, and hence any attempt to map more complicated shapes at regular intervals results in unstable parameters.

No-arbitrage models are a class of models which allow recovery of market prices of one set of securities given prices of another set. This gives a world of "relative" pricing. In a non-arbitrage-free model, securities could be priced using the model and then traded at a different price in the real world, leading to persistent profit. In simplest terms, no-arbitrage is the absence of a "free lunch".

No-arbitrage is very important in yield curve models, since most complex structures are limiting cases of simpler structures (such as swaps, caps, floors) and hence ideally the model should recover the prices of the latter exactly. Also, hedging is done using the simpler structures, so the absence of no-arbitrage would mean the accounting process would be distorted by imaginary gains and losses.

Hull describes equilibrium models as providing security prices as an output of the model, whereas no-arbitrage models take security prices as an input.

Equilibrium models are no-arbitrage as far as the stylised economy they describe is concerned, but the inadequacy of their formulation usually makes them poor at pricing complicated structures (and sometimes simple structures). Hence the possibility of arbitrage is introduced amongst instruments traded in the more complex ("real") economy.

6 (i) Value at risk (VaR) for a portfolio is a number which indicates the maximum loss which the portfolio can sustain over a given timeframe (say 1 day) for a given confidence level (say 95% or 99% certainty).

It is normally calculated by taking into account:

- future price movements (volatility)
- the interdependencies of the portfolio constituents (correlations).

If normal returns are assumed, the VaR of the portfolio can be synthesised easily from the VaR of the constituent individual holdings. If the portfolio has very many holdings, this can lead to a large correlation matrix, so some "benchmarking" is necessary to first express positions in terms of representative indices (FTSE100, FTSE250, SmallCap, AllShare etc.) before calculating VaR.

VaR is essentially a linear measure of risk, i.e. assumes the loss is always proportional to the exposure, which is true except for option-type instruments. VaR can be adjusted for these convex securities, but not if the convexity is too pronounced (e.g. very "out-of-the-money" or exotic options).

As with many such questions, a shorter answer could attract full marks. More details is given here to illustrate the range of possible points that could be covered.

(ii) In note format:

(a) **matrix**

- standard method — statistically sound
- too few factors not precise — too many and matrix too big so cumbersome to calculate
- correlations not stable or even accurate
- not forward looking — past not guide to future
- problem in choosing correct amount of history to compute vols and correlations as far past may not be relevant (could weight nearer observations but then have to choose weights!)
- normal model is the only practical one — not realistic for outliers (kurtosis effect) …
- … hence VaR does not really give the extremes — bit of a problem since it is supposed to be risk capital
- easy to calculate based on benchmarks (e.g. FTSE index) — not so easy for individual equities

(b) **backtesting (= historical simulation)**

- fast to compute
- very easy to obtain distribution of losses from the data without relying on normal model (but may be a weakness since distribution is implicit)
- no problem with correlation assessment — all implicit (again, may be a weakness)
- not forward looking
- very affected by particular major events e.g. ‘87 equity crash, ‘92 sterling fall or ‘98 credit market collapse

- allows for specific risk as well as sectoral
- problem in knowing how far back to go (as for (a))
- difficult to cope with new issues since not available throughout history …
- … so have to use sectors and hence thrown back on correlation problem/specific vs index

(c) **Monte-Carlo & GARCH time series**

Monte Carlo and GARCH are actually totally separate methodologies, but the question has linked them for convenience

- Monte Carlo — uses random number generator to provide potential values for the stochastic term of an interest-rate model — one set of random numbers gives one simulated path
- can tackle path-dependent and non-linear instruments easily
- extendible to many assets as can cope with several dimensions without excessive calculation, since number of calculations is proportional to m, not m^2 as in matrix method (m = number of factors)
- however, is slow to converge to solution (rate of $\frac{1}{\sqrt{n}}$ for n iterations), unless random numbers "tweaked" to give better coverage of domain by variance-reduction techniques
- needs an explicit underlying model, which may have inadequacies
- technically quite complicated to set up — needs skilled programming, especially to get a truly random series
- GARCH — predicts volatility and correlations based on evolution of previous observations, using an autoregressive (i.e. implicitly fitting to itself) relationship.
- good to try to look forward — VaR really should use implied vols not historical vols

- problem is we are not really sure what model we are using

- forecasting is unstable and maybe completely erroneous if trying to predict in a range outside its "experience"

- not so useful for predicting correlations — number of parameters required is enormous and problems with keeping output values in range −1 to +1.

There are many other points which could be raised. The above is only a selection.

(iii) At the linear level, one could use option delta to give exposure at current level and enter this into the VaR model.

However, non-linear characteristic of options (gamma) make VaR's linear approximation only work for small range about current market level — hence no use for capital on extreme move.

Exotic options also have very variable gamma and vega (volatility) profiles so risk profile at current level is misleading. Some exotics are virtually riskless at current levels, but "blow up" somewhere far away (e.g. barrier options or cancellation options).

Scenario testing takes a range of prices (say from −50% to +50% of current price in steps of 10%) and a range of implied volatilities (say from −20% to +20% of current volatility in steps of 10%) to give a grid of option values. … … This can then be used as a pricing tool in the backtesting model, so whatever price level was reached in the historical simulation can be translated into a portfolio P&L.

The advantage of such a method is it allows for non-linearity and also uses real values in the scenario — the problem otherwise is to assign appropriate probabilities to the grid elements.

Dowd refers to Scenario testing as Factor Push analysis in Chapter 6. He also refers to something similar called Scenario Simulation in Chapter 5, and then Scenario Analysis in Chapter 6 as a method of assessing the effect of extreme economic events on the portfolio. The syllabus is not concerned with details of Scenario Analysis, but candidates should be aware of the above mechanical process as it is enshrined in securities regulations (particularly, the Capital Adequacy Directive — see Kemp D.4.4).