9.5 Complex variables

1. Consider the function $u+iv=f(z)$ where

$$f(z)=\begin{cases}\dfrac{x^3(1-i)-y^3(1-i)}{x^2+y^2}, & z\neq 0\\ 0, & z=0\end{cases}$$

for this function two statements are as follows:

Statement 1 : $f(z)$ satisfy Cauch–Riemann equation at the origin.

Statement 2 : $f'(0)$ does not exist

The correct statement are

(A) only 1 (B) only 2

(C) Both 1 & 2 (D) neither 1 nor 2

2. If $f(z)=u+iv$, then consider the four solution for $f'(z)$

(1) $\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial x}$ (2) $\frac{\partial v}{\partial y}-i\frac{\partial v}{\partial x}$

(3) $\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}$ (4) $\frac{\partial u}{\partial y}-i\frac{\partial v}{\partial y}$

The correct solution for $f'(z)$ are

(A) 1 & 2 (B) 3 & 4

(C) 1 & 3 (D) 2 & 4

3. If $f(s)=x^2+iy^2$, then $f'(z)$ exist at all points on the line

(A) $x=y$ (B) $x=-y$

(C) $x=2+y$ (D) $y=x+2$

4. The conjugate of the function $u=2x(1-y)$ is

(A) x^2+y^2-2y+c (B) x^2-y^2+2y+c

(C) x^2-y^2-2y+c (D) None of the above

5. If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $v-v=e^x(\cos y-\sin y)$, the $f(z)$ in terms of z is

(A) $e^{-z^2}+(1+i)c$ (B) $e^{-z}+(1+i)c$

(C) $e^{z}+(1+i)c$ (D) $e^{-2z}+(1+i)c$

6. If $u=\sinh x\cos y$ then the analytic function $f(z)=u+jv$ is

(A) $\cosh^{-1} z+ic$ (B) $\cosh z+ic$

(C) $\sinh z+ic$ (D) $\sinh^{-1} z+ic$

7. If $v=2xy$, then the analytic function $f(z)=u+iv$ is

(A) z^2+c (B) $z^{-2}+c$

(C) z^3+c (D) $z^{-3}+c$

8. If $v=\frac{x-y}{x^2+y^2}$, then analytic function $f(z)=u+iv$ is

(A) $z+c$ (B) $z^{-1}+c$

(C) $(1-i)\frac{1}{z}+c$ (D) $(1+i)\frac{1}{z}+c$

9. If $u=\frac{\sin 2x}{\cosh 2y-\cos 2x}$, then the analytic function $f(z)=u+iv$ is

(A) $\cot z+ic$ (B) $\operatorname{cosec} z+ic$

(C) $\sinh z+ic$ (D) $\cosh z+ic$

10. The integration of $f(z)=x^2+ixy$ from A(1, 1) to B(2, 4) along the straight line AB joining the two points is

(A) $\frac{-29}{3}+i11$ (B) $\frac{29}{3}-i11$

(C) $\frac{23}{5}+i6$ (D) $\frac{23}{5}-i6$

11. $\int_c \frac{e^{2z}}{(z+1)^4}\,dz=?$ where c is the circle of $|z|=3$

(A) $\frac{4\pi i}{9}e^{-3}$ (B) $\frac{4\pi i}{9}e^{3}$

(C) $\frac{4\pi i}{3}e^{-1}$ (D) $\frac{8\pi i}{3}e^{-2}$

12. $\int_c \frac{1-2z}{z(z-1)(z-2)}dz = ?$ where c is the circle $|z| = 1.5$

(A) $2 + i6\pi$ (B) $4 + i3\pi$

(C) $1 + i\pi$ (D) $i3\pi$

13. $\int_c (z - z^2)dz = ?$ where c is the upper half of the circle $z = 1$

(A) $\frac{-2}{3}$ (B) $\frac{2}{3}$

(C) $\frac{3}{2}$ (D) $\frac{-3}{2}$

14. $\int_c \frac{\cos \pi z}{z-1}dz = ?$ where c is the circle $|z| = 3$

(A) $i2\pi$ (B) $-i2\pi$

(C) $i6\pi^2$ (D) $-i6\pi^2$

15. $\int_c \frac{\sin \pi z^2}{(z-2)(z-1)}dz = ?$ where c is the circle $|z| = 3$

(A) $i6\pi$ (B) $i2\pi$

(C) $i4\pi$ (D) 0

16. The value of $\frac{1}{2\pi i}\int_c \frac{\cos \pi z}{z^2-1}dz$ around a rectangle with vertices at $2 \pm i$, $-2 \pm i$ is

(A) 6 (B) $i2e$

(C) 8 (D) 0

Statement for Q. 17–18:

$f(z_0) = \int_c \frac{3z^2 + 7z + 1}{(z - z_0)}dz$, where c is the circle $x^2 + y^2 = 4$.

17. The value of $f(3)$ is

(A) 6 (B) $4i$

(C) $-4i$ (D) 0

18. The value of $f'(1 - i)$ is

(A) $7(\pi + i2)$ (B) $6(2 + i\pi)$

(C) $2\pi(5 + i13)$ (D) 0

Statement for 19–21:

Expand the given function in Taylor's series.

19. $f(z) = \frac{z-1}{z+1}$ about the points $z = 0$

(A) $1 + 2(z + z^2 + z^3......)$

(B) $-1 - 2(z - z^2 + z^3......)$

(C) $-1 + 2(z - z^2 + z^3......)$

(D) None of the above

20. $f(z) = \frac{1}{z+1}$ about $z = 1$

(A) $\frac{-1}{2}\left[1 - \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2.......\right]$

(B) $\frac{1}{2}\left[1 - \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2.......\right]$

(C) $\frac{1}{2}\left[1 + \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2.......\right]$

(D) None of the above

21. $f(z) = \sin z$ about $z = \frac{\pi}{4}$

(A) $\frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 - ....\right]$

(B) $\frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) + \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 + ...\right]$

(C) $\frac{1}{\sqrt{2}}\left[1 - \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 - ....\right]$

(D) None of the above

22. If $|z + 1| < 1$, then z^{-2} is equal to

(A) $1 + \sum_{n=1}^{\infty}(n+1)(z+1)^{n-1}$

(B) $1 + \sum_{n=1}^{\infty}(n+1)(z+1)^{n+1}$

(C) $1 + \sum_{n=1}^{\infty}n(z+1)^n$

(D) $1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$

Statement for Q. 23–25.

Expand the function $\frac{1}{(z-1)(z-2)}$ in Laurent's series for the condition given in question.

23. $1 < |z| < 2$

(A) $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + ...$

(B) $... - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{18}z^3 - ...$

(C) $\frac{1}{z^2}+\frac{3}{z^2}+\frac{7}{z^4}$ ....

(D) None of the above

24. $|z|>2$

(A) $\frac{6}{z}+\frac{13}{z^2}+\frac{20}{z^3}+$........ (B) $\frac{1}{z}+\frac{8}{z^2}+\frac{13}{z^3}+$.........

(C) $\frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+$......... (D) $\frac{2}{z^2}-\frac{3}{z^3}+\frac{4}{z^4}-$........

25. $|z|<1$

(A) $1+3z\,\frac{+7}{2}z^2+\frac{15}{4}z^2$ ....

(B) $\frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\frac{15}{16}z^3$ ....

(C) $\frac{1}{4}+\frac{3}{4}+\frac{z^2}{8}+\frac{z^3}{16}$.......

(D) None of the above

26. If $|z-1|<1$, the Laurent's series for $\frac{1}{z(z-1)(z-2)}$ is

(A) $-(z-1)-\frac{(z-1)^3}{2!}-\frac{(z-1)^5}{5!}-$...........

(B) $-(z-1)^{-1}-\frac{(z-1)^3}{2!}-\frac{(z-1)^5}{5!}-$.........

(C) $-(z-1)-(z-1)^3-(z-1)^5-$..........

(D) $-(z-1)^{-1}-(z-1)-(z-1)^3-(z-1)^5-$.........

27. The Laurent's series of $\frac{1}{z(e^z-1)}$ for $|z|<2$ is

(A) $\frac{1}{z^2}+\frac{1}{2z}+\frac{1}{12}+6z+\frac{1}{720}z^2+$..........

(B) $\frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}-\frac{1}{720}z^2+$..........

(C) $\frac{1}{z}+\frac{1}{12}+\frac{1}{634}z^2+\frac{1}{720}z^2+$..........

(D) None of the above

28. The Laurent's series of $f(z)=\frac{z}{(z^2+1)(z^2+4)}$ is, where $|z|<1$

(A) $\frac{1}{4}z-\frac{5}{16}z^3+\frac{21}{64}z^5$..........

(B) $\frac{1}{2}+\frac{1}{4}z^2+\frac{5}{16}z^4+\frac{21}{64}z^6$ ....

(C) $\frac{1}{2}z-\frac{3}{4}z^3+\frac{15}{8}z^5$.........

(D) $\frac{1}{2}+\frac{1}{2}z^2+\frac{3}{4}z^4+\frac{15}{8}z^6$ ....

29. The residue of the function $\frac{1-e^{Zz}}{z^4}$ at its pole is

(A) $\frac{4}{3}$ (B) $\frac{-4}{3}$

(C) $\frac{-2}{3}$ (D) $\frac{2}{3}$

30. The residue of $z\cos\frac{1}{z}$ at $z=0$ is

(A) $\frac{1}{2}$ (B) $\frac{-1}{2}$

(C) $\frac{1}{3}$ (D) $\frac{-1}{3}$

31. $\int_c \frac{1-2z}{z(1-z)(z-2)}dz=?$ where c is $|z|=1.5$

(A) $-i3\pi$ (B) $i3\pi$

(C) 2 (D) –2

32. $\int_c \frac{z\ \cos z}{\left(z-\frac{\pi}{2}\right)}dz=?$ where c is $|z-1|=1$

(A) 6π (B) -6π

(C) $i2\pi$ (D) None of the above

33. $\int_c z^2 e^{\frac{1}{z}}dz=?$ where c is $|z|=1$

(A) $i3\pi$ (B) $-i3\pi$

C) $\frac{i\pi}{3}$ (D) None of the above

34. $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=?$

(A) $\frac{-2\pi}{\sqrt{2}}$ (B) $\frac{2\pi}{\sqrt{3}}$

(C) $2\pi\sqrt{2}$ (D) $-2\pi\sqrt{3}$

35. $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx=?$

(A) $\frac{\pi\ ab}{a+b}$ (B) $\frac{\pi\ (a+b)}{ab}$

(C) $\frac{\pi}{a+b}$ (D) $\pi\ (a+b)$

36. $\int_0^{\infty} \frac{dx}{1+x^6} = ?$

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$

(C) $\frac{2\pi}{3}$ (D) $\frac{\pi}{3}$

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Solutions

1. (C) Since, $f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$; $z \neq 0$

$\Rightarrow \quad u = \frac{x^3 - y^3}{x^2+y^2}$; $\quad v = \frac{x^3+y^3}{x^2+y^2}$

Cauchy Riemann equations are

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

By differentiation the value of $\frac{\partial u}{\partial x}, \frac{\partial y}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at$(0,0)$

we get $\frac{0}{0}$, so we apply first principle method.

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h,\ 0) - u(0,0)}{h} = \lim_{h \to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial u}{\partial v} = \lim_{h \to 0} \frac{u(0, 0+k) - u(0,0)}{k} = \lim_{k \to 0} \frac{-k^3/k^2}{k} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0,0)}{h} = \lim_{h \to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,\ 0+k), v(0,0)}{k} = \lim_{k \to 0} \frac{k^3/k^2}{k} = 1$$

Thus, we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, Cauchy-Riemann equations are satisfied at $z = 0$.

Again, $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$

$$= \lim_{z \to 0} \left[\frac{(x^3 - y^3) + i(x^3+y^3)}{(x^2+y^2)} \frac{1}{(x+iy)} \right]$$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{z \to 0} \left[\frac{(x^3 - y^3) + i(x^3+y^3)}{(x^2+y^2)} \frac{1}{(x+iy)} \right]$$

$$= \frac{2i}{2(1+i)} = \frac{1+i}{2}$$

Again let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{x \to 0} \left[\frac{x^3 + i(x^3)}{(x^2)} \frac{1}{x} \right] = 1 + i$$

So we see that $f'(0)$ is not unique. Hence $f'(0)$ does not exist.

2. (A) Since, $f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$

Or $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$ ....(1)

Now, the derivative $f'(z)$ exits of the limit in equation (1) is unique i.e. it does not depends on the path along which $\Delta z \to 0$.

Let $\Delta z \to 0$ along a path parallel to real axi

$\Rightarrow \quad \Delta y = 0 \therefore \Delta z \to 0 \quad \Rightarrow \quad \Delta x \to 0$

Now equation (1)

$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + i \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad ....(2)$$

Again, let $\Delta z \to 0$ along a path parallel to imaginary axis, then $\Delta x \to 0$ and $\Delta z \to 0 \to \Delta y \to 0$

Thus from equation (1)

$$\phi'(z) = \lim_{\Delta y \to 0} \frac{\Delta z + i\Delta v}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta u}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{\Delta v}{i\Delta z} = \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = \frac{-i\partial u}{\partial y} + \frac{\partial v}{\partial y} ....(3)$$

Now, for existence of $f'(z)$ R.H.S. of equation (2) and (3) must be same i.e.,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

3. (A) Given $f(z) = x^2 + iy^2$ since, $f(z) = u + iv$

Here $u = x^2$ and $v = y^2$

Now, $u = x^2 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = 0$

and $v = y^2 \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 2y$

we know that

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} ....(1)$$

and $f'(z)=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$....(2)

Now, equation (1) gives $f'(z)=2x$ ....(3)

and equation (2) gives $f'(z)=2y$ ....(4)

Now, for existence of $f'(z)$ at any point is necessary that the value of $f'(z)$ most be unique at that point, whatever be the path of reaching at that point

From equation (3) and (4) $2x=2y$

Hence, $f'(z)$ exists for all points lie on the line $x=y$.

4. (B) $\frac{\partial u}{\partial x}=2(1-y)$; $\frac{\partial^2 u}{\partial x^2}=0$ ....(1)

$\frac{\partial u}{\partial y}=-2x$; $\frac{\partial^2 u}{\partial y^2}=0$ ....(2)

$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, Thus u is harmonic.

Now let v be the conjugate of u then

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

(by Cauchy-Riemann equation)

$\Rightarrow \quad dv=2x\,dx+2(1-y)dy$

On integrating $v=x^2-y^2+2y+C$

5. (C) Given $f(z)=u+i\,v$ ....(1)

$\Rightarrow \quad if(z)=-v+iu$ ....(2)

add equation (1) and (2)

$\Rightarrow \quad (1+i)f(z)=(u-v)+i(u+v)$

$\Rightarrow \quad F(z)=U+iV$

where, $F(z)=(1+i)f(z)$; $U=(u-v)$; $V=u+v$

Let $F(z)$ be an analytic function.

Now, $U=u-v=e^x(\cos y-\sin y)$

$\frac{\partial U}{\partial x}=e^x(\cos y-\sin y)$

and $\frac{\partial U}{\partial y}=e^x(-\sin y-\cos y)$

Now, $dV=\frac{-\partial U}{\partial y}dx+\frac{\partial U}{\partial x}dy$....(3)

$=e^x(\sin y+\cos y)dx+e^x(\cos y-\sin y)dy$

$=d[e^x(\sin y+\cos y)]$

on integrating $V=e^x(\sin y+\cos y)+c_1$

$F(z)=U+iV=e^x(\cos y-\sin y)+ie^x(\sin y+\cos y)+ic_1$

$=e^x(\cos y+i\sin y)+ie^x(\cos y+i\sin y)+ic_1$

$F(z)=(1+i)e^{x+iy}+ic_1=(1+i)e^z+ic_1$

$(1+i)f(z)=(1+i)e^z+ic_1$

$\Rightarrow \quad f(z)=e^z+\frac{i}{1+i}c_1=e^z+c_1\frac{i(1-i)}{(1+i)(1-i)}$

$=e^z+\frac{(i+1)}{2}c_1$

$\Rightarrow \quad f(z)=e^z+(1+i)c$

6. (C) $u=\sinh x\cos y$

$\frac{\partial u}{\partial x}=\cosh x\cos y=\phi(x,y)$

and $\frac{\partial u}{\partial y}=-\sinh x\sin y=\psi(x,y)$

by Milne's Method

$f'(z)=\phi(z,0)-i\psi(z,0)=\cosh z-i\cdot 0=\cosh z$

On integrating $f(z)=\sinh z+$ constant

$\Rightarrow \quad f(z)=w=\sinh z+ic$

(As u does not contain any constant, the constant c is in the function x and hence i.e. in w).

7. (A) $\frac{\partial v}{\partial x}=2y=h(x,y)$, $\frac{\partial v}{\partial y}=2x=g(x,y)$

by Milne's Method

$f'(z)=g(z,0)+ih(z,0)=2z+i\,0=2z$

On integrating $f(z)=z^2+c$

8. (D) $\frac{\partial v}{\partial y}=\frac{-(x^2+y^2)-(x-y)2y}{(x^2+y^2)^2}$

$=\frac{y^2-x^2-2xy}{(x^2+y^2)^2}=g(x,y)$

$\frac{\partial v}{\partial x}=\frac{(x^2+y^2)-(x-y)2x}{(x^2+y^2)^2}=\frac{y^2-x^2+2xy}{(x^2+y^2)^2}=h(x,y)$

By Milne's Method

$f'(z)=g(z,0)+ih(z,0)=-\frac{1}{z^2}+i\left(-\frac{1}{z^2}\right)=-(1+i)\frac{1}{z^2}$

On integrating

$f(z)=(1+i)\int\frac{1}{z^2}dz+c=(1+i)\frac{1}{z}+c$

9. (A) $\frac{\partial u}{\partial x}=\frac{2\cos 2x(\cosh 2y-\cos 2x)-2\sin^2 2x}{(\cosh 2y-\cos 2x)^2}$

$=\frac{2\cos 2x\cosh 2y-2}{(\cosh 2y-\cos 2y)^2}=\phi(x,y)$

$\frac{\partial u}{\partial y}=\frac{2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2}=\psi(x,y)$

By Milne's Method

$f'(z)=\phi(z,0)-i\psi(z,0)$

$=\frac{2\cos 2z-2}{(1-\cos 2z)^2}-i(0)=\frac{-2}{1-\cos 2z}=-\text{cosec}^2 z$

On integrating

$f(z) = -\int \operatorname{cosec}^2 z\, dz + ic = \cot z + ic$

10. $x = at + b,\ y = ct + d$

On A, $z = 1 + i$ and On B, $z = 2 + 4i$

Let $z = 1 + i$ corresponds to $t = 0$

and $z = 2 + 4i$ corresponding to $t = 1$

then, $t = 0 \quad \Rightarrow \quad x = b,\ y = d$

$\Rightarrow \quad b = 1,\ d = 1$

and $t = 1 \quad \Rightarrow \quad x = a + b,\ y = c + d$

$\Rightarrow \quad 2 = a + 1,\ 4 = c + 1 \Rightarrow \quad a = 1,\ c = 3$

AB is , $y = 3t + 1 \Rightarrow \quad dx = dt$; $dy = 3\, dt$

$\int_c f(z)dz = \int_c (x^2 + ixy)(dx + idy)$

$= \int_{t=0}^{1} [(t+1)^2 + i(t+1)(3t+1)][dt + 3i\, dt]$

$= \int_0^1 [(t^2 + 2t + 1) + i(3t^2 + 4t + 1)](1 + 3i)dt$

$= (1+3i)\left[\frac{t^3}{3} + t^2 + t + i(t^3 + 2t^2 + t)\right]_0^1 = -\frac{29}{3} + 11i$

11. (D) We know by the derivative of an analytic function that

$f''(z_o) = \frac{n!}{2\pi i}\int_c \frac{f(z)\, dz}{(z - z_o)^{n+1}}$

Or $\int_c \frac{f(z)\, dz}{(z - z_o)^{n+1}} = \frac{2\pi i}{n!} f^n(z_o)$

Taking $n = 3$, $\int_c \frac{f(z)\, dz}{(z - z_o)^4} = \frac{\pi i}{3} f''(z_o)$ ....(1)

Given $f_c \frac{e^{2z}dz}{(z+1)^4} = \int_c \frac{e^{2z}dz}{[z - (-1)]^4}$

Taking $f(z) = e^{2z}$, and $z_o = -1$ in (1), we have

$\int_c \frac{e^{2z}dz}{(z+1)^4} = \frac{\pi i}{3} f'''(-1)$....(2)

Now, $f(z) = e^{2z} \quad \Rightarrow \quad f'''(z) = 8e^{2z}$

$\Rightarrow \quad f'''(-1) = 8e^{-2}$

equation (2) have

$\Rightarrow \quad \int_c \frac{e^{2z}dz}{(z+1)^4} = \frac{8\pi i}{3} e^{-2}$ ....(3)

If is the circle $|z| = 3$

Since, $f(z)$ is analytic within and on $|z| = 3$

$\int_{|z|=3} \frac{e^{2z}dz}{(z+1)^4} = \frac{8\pi i}{3} e^{-z}$

12. (D) Since, $\frac{1 - 2z}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{1}{z-1} - \frac{3}{2(z-2)}$

$\int_c \frac{1 - 2z}{z(z-1)(z-2)} dz = \frac{1}{2} I_1 + I_2 - \frac{3}{2} I_3$....(1)

Since, $z = 0$ is the only singularity for $I_1 = \int_c \frac{1}{z} dz$ and it lies inside $|z| = 1.5$, therefore by Cauchy's integral Formula

$I_1 = \int_c \frac{1}{z}\, dz = 2\pi i$ ....(2)

$\left[f(z_o) = \frac{1}{2\pi i}\int_c \frac{f(z)\, dz}{z - z_o}\right]$ [Here $f(z) = 1 = f(z_o)$ and $z_o = 0$]

Similarly, for $I_2 = \int_c \frac{1}{z-1} dz$, the singular point $z = 1$ lies inside $|z| = 1.5$, therefore $I_2 = 2\pi i$....(3)

For $I_3 = \int_c \frac{1}{z-2}\, dz$, the singular point $z = 2$ lies outside the circle $|z| = 1.5$, so the function $f(z)$ is analytic everywhere in c i.e. $|z| = 1.5$, hence by Cauchy's integral theorem

$I_3 = \int_c \frac{1}{z-2}\, dz = 0$....(4)

using equations (2), (3), (4) in (1), we get

$\int_c \frac{1 - 2z}{z(z-1)(z-2)} dz = \frac{1}{2}(2\pi i) + 2\pi i - \frac{3}{2}(0) = 3\pi i$

13. (B) Given contour c is the circle $|z| = 1$

$\Rightarrow \quad z = e^{i\theta} \quad \Rightarrow \quad dz = ie^{i\theta}d\theta$

Now, for upper half of the circle, $0 \le \theta \le \pi$

$\int_c (z - z^2)dz = \int_{\theta=0}^{\pi} (e^{i\theta} - e^{2i\theta})ie^{i\theta}d\theta$

$= i\int_0^{\pi} (e^{2i\theta} - e^{3i\theta})d\theta = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi}$

$= i \cdot \frac{1}{i}\left[\frac{1}{2}\cdot(e^{2\pi i} - 1) - \frac{1}{3}(e^{3\pi x} - 1)\right] = \frac{2}{3}$

14. (B) Let $f(z) = \cos \pi z$ then $f(z)$ is analytic within and on $|z| = 3$, now by Cauchy's integral formula

$f(z_o) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - z_o}\, dz \Rightarrow \quad \int_c \frac{f(z)\, dz}{z - z_o} = 2\pi i f(z_o)$

take $f(z) = \cos \pi z$, $z_o = 1$, we have

$\int_{|z|=3} \frac{\cos \pi z}{z - 1}\, dz = 2\pi i f(1) = 2\pi i \cos \pi = -2\pi i$

15. (D) $\int_c \frac{\sin \pi z^2}{(z-1)(z-2)} dz$

$= \int_c \frac{\sin \pi z^2}{z-2} dz - \int_c \frac{\sin \pi z^2}{z-1} dz$

$= 2\pi i f(2) - 2\pi i f(1)$ since, $f(z) = \sin \pi z^2$

$\Rightarrow \quad f(2) = \sin 4\pi = 0$ and $f(1) = \sin \pi = 0$

16. (D) Let, $I = \frac{1}{2\pi i} \int_c \frac{1}{z^2-1} \cos \pi z \, dz$

$= \frac{1}{2 \cdot 2\pi i} \int_c \left(\frac{1}{z-1} - \frac{1}{z+1} \right) \cos \pi z \, dz$

Or $I = \frac{1}{4\pi i} \int_c \left(\frac{\cos nz}{z-1} - \frac{\cos nz}{z+1} \right) dz$

17. (D) $f(3) = \int_c \frac{3z^2+7z+1}{z-3} dz$, since $z_o = 3$ is the only singular point of $\frac{3z^2+7z+1}{z-3}$ and it lies outside the circle $x^2+y^2=4$ i.e., $|z|=2$, therefore $\frac{3z^2+7z+1}{z-3}$ is analytic everywhere within c.

Hence by Cauchy's theorem—

$f(3) = \int_c \frac{3z^2+7z+1}{z-3} dz = 0$

18. (C) The point $(1-i)$ lies within circle $|z|=2$ (... the distance of $1-i$ i.e., (1, 1) from the origin is $\sqrt{2}$ which is less than 2, the radius of the circle).

Let $\phi(z) = 3z^2+7z+1$ then by Cauchy's integral formula

$\int_c \frac{3z^2+7z+1}{z-z_o} dz = 2\pi i \phi(z_o)$

$\Rightarrow \quad f(z_o) = 2\pi i \phi(z_o) \quad \Rightarrow \quad f'(z_o) = 2\pi i \phi'(z_o)$

and $f''(z_o) = 2\pi i \, \phi''(z_o)$

since, $\phi(z) = 3z^2+7z+1$

$\Rightarrow \quad \phi'(z) = 6z+7$ and $\phi''(z) = 6$

$f'(1-i) = 2\pi i[6(1-i)+7] = 2\pi \, (5+13i)$

19. (C) $f(z) = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$

$\Rightarrow \quad f(0) = -1, \; f(1) = 0$

$\Rightarrow \quad f'(z) = \frac{2}{(z+1)^2} \quad \Rightarrow \quad f'(0) = 2;$

$f''(z) = \frac{-4}{(z-1)^3} \quad \Rightarrow \quad f''(0) = -4;$

$f'''(z) = \frac{12}{(z+1)^4} \quad \Rightarrow \quad f'''(0) = 12;$ and so on.

Now, Taylor series is given by

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0) + ...$$

about $z = 0$

$f(z) = -1 + z(2) + \frac{z^2}{2!}(-4) + \frac{z^3}{3!}(12) + ...$

$= -1 + 2z - 2z^2 + 2z^3 ....$

$f(z) = -1 + 2(z - z^2 + z^3 \, ...)$

20. (B) $f(z) = \frac{1}{z+1} \quad \Rightarrow \quad f(1) = \frac{1}{2}$

$f'(z) = \frac{-1}{(z+1)^2} \quad \Rightarrow \quad f'(1) = \frac{-1}{4}$

$f''(z) = \frac{2}{(z+1)^3} \quad \Rightarrow \quad f''(1) = \frac{1}{4}$

$f'''(z) = \frac{-6}{(z+1)^4} \quad \Rightarrow \quad f'''(1) = -\frac{3}{8}$ and so on.

Taylor series is

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0) + ...$$

about $z = 1$

$f(z) = \frac{1}{2} + (z-1)\left(\frac{-1}{4}\right) + \frac{(z-1)^2}{2!}\left(\frac{1}{4}\right) + \frac{(z-1)^3}{3!}\left(-\frac{3}{8}\right) + ...$

$= \frac{1}{2} - \frac{1}{2^2}(z-1) + \frac{1}{2^3}(z-1)^2 - \frac{1}{2^4}(z-1)^3 + ...$

or $f(z) = \frac{1}{2}\left[1 - \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2 - \frac{1}{2^3}(z-1)^3 + ...\right]$

21. (A) $f(z) = \sin z \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z \quad \Rightarrow \quad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and so on.

Taylor series is given by

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0) + ...$$

about $z = \frac{\pi}{4}$

$$f(z)=\frac{1}{\sqrt{2}}+\left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)+\frac{\left(z-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right)+\ldots$$

$$f(z)=\frac{1}{\sqrt{2}}\left[1+\left(z-\frac{\pi}{4}\right)-\frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(z-\frac{\pi}{4}\right)^3-\ldots\right]$$

22. (D) Let $f(z)=z^{-2}=\frac{1}{z^2}=\frac{1}{[1-(1+z)]^2}$

$f(z)=[1-(1+z)]^{-2}$

Since, $|1+z|<1$, so by expanding R.H.S. by binomial theorem, we get

$$f(z)=1+2(1+z)+3(1+z)^2+4(1+z)^3+\ldots+(n+1)(1+z)^n+\ldots$$

or $f(z)=z^{-2}=1+\sum_{n=1}^{\infty}(n+1)(z+1)^n$

23. (B) Here $f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$....(1)

Since, $|z|>1 \Rightarrow \frac{1}{|z|}<1$ and $|z|<2$

$\Rightarrow \frac{|z|}{2}<1$

$$\frac{1}{z-1}=\frac{1}{z\left(1-\frac{1}{z}\right)}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right)$$

and $\frac{1}{z-2}=\frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1}=-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{9}+\ldots\right]$

equation (1) gives—

$$f(z)=-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{9}+..\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right)$$

or $f(z)=\ldots-z^{-4}-z^{-2}-z^{-1}-\frac{1}{2}-\frac{1}{4}z-\frac{1}{8}z^2-\frac{1}{18}z^3-\ldots$

24. (C) $\frac{2}{|z|}<1 \Rightarrow \frac{1}{|z|}<\frac{1}{2}<1 \Rightarrow \frac{1}{|z|}<1$

$$\frac{1}{z-1}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right)$$

and $\frac{1}{z-2}=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots.\right)$

Laurent's series is given by

$$f(z)=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{98}{z^3}+..\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+..\right)$$

$$=\frac{1}{z}\left(\frac{1}{z}+\frac{3}{z^2}+\frac{7}{z^3}+\ldots\right)$$

$\Rightarrow f(z)=\frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\ldots$

25. (B) $|z|<1$, $\frac{1}{z-2}-\frac{1}{z-1}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1-z)^{-1}$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\right]+(1+z+z^2+z^3+\ldots)$$

$$f(z)=\frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\frac{15}{16}z^3+\ldots$$

26. (D) Since, $\frac{1}{z(z-1)(z-2)}=\frac{1}{2z}-\frac{1}{z-1}+\frac{1}{2(z-2)}$

For $|z-1|<1$ Let $z-1=u$

$\Rightarrow z=u+1$ and $|u|<1$

$$\frac{1}{z(z-1)(z-2)}=\frac{1}{2z}-\frac{1}{z-1}+\frac{1}{2(z-2)}$$

$$=\frac{1}{2(u+1)}-\frac{1}{u}+\frac{1}{2(u-1)}=\frac{1}{2}(1+u)^{-1}-u^{-1}-\frac{1}{2}(1-u)^{-1}$$

$$=\frac{1}{2}[1-u+u^2-u^3+\ldots]-u^{-1}-\frac{1}{2}(1+u+u^2+u^3+\ldots)$$

$$=\frac{1}{2}(-2u-2u^3-\ldots)-u^{-1}=-u-u^3-u^5-\ldots-u^{-1}$$

Required Laurent's series is

$$f(z)=-(z-1)^{-1}-(z-1)-(z-1)^3-(z-1)^5-\ldots$$

27. (B) Let $f(z)=\frac{1}{z(e^z-1)}$

$$=\frac{1}{z\left[1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots-1\right]}$$

$$=\frac{1}{z^2\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\ldots\right)}$$

$$=\frac{1}{z^2}\left(1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\frac{z^4}{120}+\ldots\right)^{-1}$$

$$=\frac{1}{z^2}\left[1-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\frac{z^4}{120}+\ldots\right)+\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\ldots\right)^2-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\ldots\right)^3\right.$$

$$=\frac{1}{z^2}\left[1-\frac{z}{2}-\frac{z^2}{6}-\frac{z^3}{24}-\frac{z^4}{120}+\frac{z^2}{4}+\frac{z^4}{36}+\frac{z^3}{6}+\frac{z^4}{24}-\frac{z^3}{8}-\frac{z^4}{8}-\frac{z^4}{16}\ldots\right]$$

or $f(z)=\frac{1}{z^2}\left[1-z\left(\frac{1}{2}\right)+z^2\left(-\frac{1}{6}+\frac{1}{4}\right)\ldots\right.$

$$+z^3\left(-\frac{1}{24}+\frac{1}{6}-\frac{1}{8}\right)\ldots$$

$$+z^4\left(-\frac{1}{120}+\frac{1}{36}+\frac{1}{24}-\frac{1}{8}+\frac{1}{16}\right)+\ldots\Bigg]$$

$$=\frac{1}{z^2}\left[1-\frac{1}{2}z+\frac{1}{12}z^2+0\,z^3+z^4\left(-\frac{1}{720}\right)+\ldots\right]$$

Required Laurent's series is

$$f(z)=\frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}+0.z-\frac{1}{720}z^2+\ldots$$

28. (A) Since, $f(z)=\frac{z}{(z^2+1)(z^2+4)}$

$$=\frac{z}{3(z^2+1)}-\frac{z}{3(z^2+4)}$$

$|z|<1 \quad\Rightarrow\quad |z^2|<1$

$$f(z)=\frac{z}{3}(1+z^2)^{-1}-\frac{z}{12}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$=\frac{z}{3}(1-z^2+z^4-\ldots)-\frac{z}{12}\left(1-\frac{z^2}{4}+\frac{z^4}{16}-\ldots\right)$$

or $f(z)=\frac{1}{4}z-\frac{5}{16}z^3+\frac{21}{64}z^5\ldots$

29. (B) Let $f(z)=\frac{1-e^{2z}}{z^4}$ then $f(z)$ has a pole at $z=0$ of order 4.

Residue of $f(z)$ at $z=0$

$$=\frac{1}{(m-1)!}\lim_{z\to z_0},\ \frac{d^{m-1}}{dz^{m-1}}(z-z_o)^m f(z)$$

$$=\frac{1}{(4-1)!}\lim_{z\to 0}\frac{d^3}{dz^3}\left[z^4\cdot\left(\frac{1-e^{2z}}{z^4}\right)\right]$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})=\frac{-1}{3!}\lim_{z\to 0}8e^{cz}$$

$$=\frac{-8}{6}(e^o)=\frac{-8}{6}=\frac{-4}{3}$$

30. (B) Put $z=0+t$, $f(z)=z\cos\frac{1}{z}$

$$=t\cos\frac{1}{t}=t\left(1-\frac{1}{2!}\ \frac{1}{t^2}+\frac{1}{4!}\ \frac{1}{t^4}-\ldots\right)$$

$$=t-\frac{1}{2t}+\frac{1}{24t^3}-\ldots$$

Residue of $f(z)$ at $z=0$ is the coefficient of $\frac{1}{t}$ i.e. $-\frac{1}{2}$

31. Poles of $f(z)$ are at $z=0,\ 1,\ 2$ since 0 and 1 lie within c and $c=2$ does not inside c.

$\int_c f(z)dz=2\pi i$[sum of residues at $z=0$ and at $z=1$]....(1)

Now, Residue at $z=0$ is

$$=\lim_{z\to 0} zf(z)=\lim_{z\to 0}\frac{1-2z}{(1-z)(z-2)}=\frac{1}{2}$$

and Residue at $z=1$ is

$$=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}\frac{2z-1}{z(z-2)}=-1$$

equation (1) gives

$$\int_c f(z)=2\pi i\times\left(-\frac{1}{2}-1\right)=-3\pi i$$

32. (D) $f(z)=\frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^2}$ then $f(z)$ has a pole at $z=\frac{\pi}{2}$ of order 2.

by Cauchy's residue theorem

$$\int_c f(z)dz=2\pi i\times\left(\text{Residue at } z=\frac{\pi}{2}\right)$$

Now, Residue at $z=\frac{\pi}{2}$ is

$$=\lim_{z\to\frac{\pi}{2}}\frac{d}{dz}\left[\left(z-\frac{\pi}{2}\right)^2 f(z)\right]=\lim_{z\to\frac{\pi}{2}}\frac{d}{dz}(z\cos z)$$

$$=\lim_{z\to\frac{\pi}{2}}[\cos z-z\sin z]=-\frac{\pi}{2}$$

$$\int_c f(z)dz=2\pi i\times\left(-\frac{\pi}{2}\right)=-\pi^2 i$$

33. (C) $f(z)=z^2e^{1/z}=z^2\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots\right)$

$$=z^2+z^2+\frac{1}{2}+\frac{1}{6z}+\ldots$$

The only pole of $f(z)$ is at $z=0$, which lies within the circle $|z|=1$

$$\int_c f(z)dz=2\pi i(\text{residue at } z=0)$$

Now, residue of $f(z)$ at $z=0$ is the coefficient of $\frac{1}{z}$ i.e. $\frac{1}{6}$

$$\int_c f(z)dz=2\pi i\times\frac{1}{6}=\frac{1}{3}\pi i$$

34. (B) Let $z=e^{i\theta}\quad\Rightarrow\quad d\theta=\frac{-idz}{z}\ ;\ z\le\theta\le 2\pi$

and $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=\int_c\frac{\frac{-idz}{z}}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)};\quad c:|z|=1$$

$$=-2i\int_c\frac{dz}{z^2+4z+1}$$

Let $f(z)=\dfrac{1}{z^2+4z+1}$

$f(z)$ has poles at $z=-2+\sqrt{3},\,-2-\sqrt{3}$ out of these only $z=-2+\sqrt{3}$ lies inside the circle $c:|z|=1$

$$\int_c f(z)dz=2\pi i(\text{Residue at } z=-2+\sqrt{3})$$

Now, residue at $z=-2+\sqrt{3}$

$$=\lim_{z\to -2+\sqrt{3}}(z+2-\sqrt{3})f(z)$$

$$=\lim_{z\to -2+\sqrt{3}}\frac{1}{(z+2+\sqrt{3})}=\frac{1}{2\sqrt{3}}$$

$$\int_c f(z)dz=2\pi i\times\frac{1}{2\sqrt{3}}=\frac{\pi i}{\sqrt{3}}$$

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=-2i\times\frac{\pi i}{\sqrt{3}}=\frac{2\pi}{\sqrt{3}}$$

35. (C) $I=\int_c\dfrac{z^2}{(z^2+a^2)(z^2+b^2)}dz=\int_c f(z)dz$

where c is be semi circle r with segment on real axis from $-R$ to R.

The poles are $z=\pm ia$, $z=\pm ib$. Here only $z=ia$ and $z=ib$ lie within the contour c

$$\int_c f(z)dz=2\pi i$$

(sum of residues at $z=ia$ and $z=ib$)

Residue at $z=ia$,

$$=\lim_{z\to ia}(z-ia)\frac{z^2}{(z-ia)(z-ia)(z^2+b^2)}=\frac{a}{2i(a^2-b^2)}$$

Residue at $z=ib$

$$=\lim_{z\to ib}(z-ib)\frac{z^2}{(z-ia)(z+ia)(z+ib)(z-ib)}=\frac{-b}{2i(a^2-b^2)}$$

$$\int_c f(z)dz=\int_r f(z)dz+\int_{-R}^{R}f(z)dz$$

$$=\frac{2\pi i}{2i\,(a^2-b^2)}(a-b)=\frac{\pi}{a+b}$$

Now $\displaystyle\int_r f(z)dz=\int_0^{\pi}\frac{ie^{2i\theta}iRe^{i\theta}d\theta}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)}$

$$=\int_0^{\pi}\frac{\frac{e^{3i\theta}}{R}d\theta}{\left(e^{2i\theta}+\frac{a^2}{R^2}\right)\left(e^{2i\theta}+\frac{b^2}{R^2}\right)}$$

Now when $R\to\infty$, $\int_r b(z)dz=0$

$$\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dz=\frac{\pi}{a+b}$$

36. (C) Let $I=\int_c\dfrac{dz}{1+z^6}=\int_c f(z)dz$

c is the contour containing semi circle r of radius R and segment from $-R$ to R.

For poles of $f(z)$, $1+z^6=0$

$\Rightarrow\quad z=(-1)^{\pi/6}=e^{i(2n+1)\pi/6}$

where $n=0,\,1,\,2,\,3,\,4,\,5,\,6$

Only poles

$z=\dfrac{-\sqrt{3}+i}{2}$, i, $\dfrac{\sqrt{3}+i}{2}$ lie in the contour

Residue at $z=\dfrac{+\sqrt{3}+i}{2}$

$$=\frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_1-z_5)(z_1-z_6)}$$

$$=\frac{1}{3i(1+\sqrt{3}\,i)}=\frac{1-\sqrt{3}i}{12i}$$

Residue at $z=i$ is $\dfrac{1}{6i}$

Residue at $z=\dfrac{1+\sqrt{3}i}{12i}$ is

$$=\frac{1}{3i(1-\sqrt{3}i)}=\frac{1+\sqrt{3}i}{12i}$$

$$\int_c f(z)dz=\int_r f(z)dz+\int_{-R}^{R}f(z)dz$$

$$=\frac{2\pi i}{12i}(1-\sqrt{3}i+1+\sqrt{3}i+2i)=\frac{2\pi}{3}$$

or $\displaystyle\int_r f(z)dz+\int_{-R}^{R}f(z)dz=\frac{2\pi}{3}$....(1)

Now $\int_c f(z)dz$

$$=\int_0^{\pi}\frac{iRe^{i\theta}d\theta}{1+R^6e^{6i\theta}}=\int_0^{\pi}\frac{\frac{ie^{i\theta}d\theta}{R^5}}{\frac{1}{R^6}+e^{6i\theta}}$$

where $R\to\infty$, $\int_r f(z)dz\to 0$

$$(1)\to\int_0^{\infty}\frac{ax}{1+x^6}=\frac{2\pi}{3}$$

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