ENGINEERING MATHEMATICS—2008

SEMESTER - 2

Time : 3 Hours *Full Marks : 70*

GROUP-A

(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following : 10 x 1 = 10

(i) $\frac{1}{D-1}x^2$ is equal to

(a) x^2+2x+2 (b) $-(x^2+2x+2)$ (c) $2x-x^2$ (d) $-(2x-x^2)$

(ii) An integrating factor for the differential equation $\frac{dy}{dt}+y=1$ is

(a) e^t (b) $\frac{e}{t}$ (c) et (d) $\frac{t}{e}$

(iii) The general solution of the differential equation $D^2y+9y=0$ is

(a) $Ae^{3x}+Be^{-3x}$ (b) $(A+Bx)e^{3x}$

(c) $A\cos 3x+B\sin 3x$ (d) $(A+Bx)\sin 3x$.

(iv) If $L\{f(t)\}=\tan^{-1}\left(\frac{1}{P}\right)$, then $L\{tf(t)\}$ is

(a) $\tan^{-1}\left(\frac{1}{P^2}\right)$ (b) $\frac{1}{1+P^2}$ (c) $\frac{1}{1+P}$ (d) $\tan^{-1}\left(\frac{2}{\pi P}\right)$

(v) In Simpson's 1/3 rule of finding $\int_a^b f(x)\,dx$, $f(x)$ is approximated by

(a) line segment (b) parabola

(c) circular sector (d) parts of ellipse.

(vi) The system of equations $x+2y=5$, $2x+4y=7$ has

(a) unique solution (b) no solution

(c) infinite number of solutions (d) none of these.

(vii) The four vectors (1, 1, 0, 0), (1, 0, 0. 1), (1, 0, a, 0), (0, 1, a,b) are linearly independent if.

(a) $a\neq 0, b\neq 2$ (c) $a\neq 0, b\neq -2$ (b) $a\neq 2, b\neq 0$ (d) $a\neq -2, b\neq 0$.

(viii) The rank of a matrix $A=\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}$ is

(a) 1 (b) 2 (c) 3 (d) none of these.

(ix) The matrix $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is

(a) an orthogonal matrix (b) a symmetric matrix

(c) an idempotent matrix (d) a null matrix.

(x) $T : R^2 \to R^2$ is defined by $T(x_1, x_2) = (2x_1 - x_2, x_1 - x_2)$. Then kernel of T is

(a) $\{(1, 2)\}$ (b) $\{(1, -1)\}$

(c) $\{(0, 0)\}$ (d) $\{(1, 2), (1, -1)\}$

(xi) $\Delta^2 e^x$ is equal to (where h = 1)

(a) $(e-1)^2 e^x$ (b) $(e-1)e^x$ (c) $e^{2x}(e-1)$ (d) e^{2x+1}

(xii) The eigenvalue of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ is

(a) 0,0, 1 (b) 1, 2, 3 (b) 2, 3, 6 (d) none of these.

(xiii) The mapping $f : R \to R$ where $f(x) = \sin x$, $x \in R$ is

(a) one-one (b) onto

(c) neither one-one nor onto (d) both one-one and onto.

1. (i) $\frac{1}{(D-1)}x^2 = -(1-D)^{-1}x^2 = -(1+D+D^2+..)x^2 = -(x^2+2x+2)$

(ii) $I.F = e^{\int dt} = e^t$

(iii) $D^2y + 9y = 0$; Let $y = e^{mx}$ is the trial Solution.

$\therefore$ The A.E. is $m^2+9=0 \Rightarrow m^2 = -9 \Rightarrow m = \pm 3i$

$\therefore$ The general Solution is $y = A\cos 3x + B\sin 3x$

(iv) $L\{f(t)\} = \tan^{-1}\left(\frac{1}{p}\right) \therefore L\{tf(t)\} = (-1)\frac{d}{dp}\left[\tan^{-1}\left(\frac{1}{p}\right)\right] = -\frac{-\frac{1}{p^2}}{1+\frac{1}{p^2}} = \frac{1}{1+p^2}$

(v) b – parabola

(vi) $\begin{pmatrix} 1 & 2 & : & 5 \\ 2 & 4 & : & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & -3 \end{pmatrix}$

The rank of the coefficient matrix is 1.

The rank of the augmented matrix is 2

$\therefore$ The System has no solution.

(vii) $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & a & b \end{pmatrix} \det(A) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & a & 0 \\ 0 & a & a & b \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & a & -1 \\ 0 & 0 & a & 1+b \end{vmatrix}$

$= -1\,(a + ab + a) = -a\,(b+2)$

$\therefore |A| \neq 0 \Rightarrow a \neq 0$ and $b \neq -2$

(viii) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \therefore \text{Rank} = 2.$

(ix) $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, A^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$

$A \cdot A^T = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore A^{-1} = A^T$ $\quad\therefore$ The matrix is orthogonal

(x) $T(x_1, x_2) = \theta \Rightarrow 2x_1 - x_2 = 0$ & $x_1 + x_2 = 0$
Solve $x_1 = 0, x_2 = 0 \therefore \ker(T) = \{(0,0)\}$

(xi) $\Delta^2 e^x = \Delta(\Delta e^x) = \Delta(e^{x+h} - e^x) = \Delta[e^x \cdot (e^h - 1)]$

$= (e^h - 1)\Delta e^x = (e^h - 1)^2 e^x$

$= (e-1)^2 e^x$

(xii) $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$

$\Rightarrow \lambda = 1,2,3$

(xiii) $f(x) = \sin x$,
The mapping $f : R \rightarrow R$.
we know $-1 \leq \sin x \leq 1$

So, $f(R) = [-1,1] \neq R$.

$\therefore$ It is not onto.

Let $\text{Sin } x_1 = \text{Sin } x_2$

$\Rightarrow \sin(2\pi + x_1) = \sin x_2$

$\Rightarrow x_1 = x_2 - 2\pi$ $\Rightarrow x_1 \neq x_1$

$\therefore$ It is not one to one.

$\therefore$ The mapping neither one to one nor onto.

GROUP – B

(Short Answer Type Questions)

Answer *any three* of the following. $3 \times 5 = 1$

2. Solve the differential equation by Laplace Transformation.

$$\frac{d^2y}{dt^2} + 9y = 1 \qquad y(0) = 1,\ y\left(\frac{\pi}{2}\right) = -1$$

Ans. $\frac{d^2y}{dt^2} + 9y = 1$

Taking leplace transfrom both sides, we get.

$S^2L(y) - S\,y(0) - y'(0) + 9L(y) = L(1)$

Let $y'(0) = C$

$\therefore S^2L(y) - S - C + 9L(y) = \frac{1}{S}$

$$\Rightarrow L(y) = \frac{1}{S(S^2+9)} + \frac{S+C}{S^2+9} = \frac{1}{9}\left(\frac{1}{S} - \frac{S}{S^2+9}\right) + \frac{S+C}{S^2+9}$$

$$= \frac{1}{9}\cdot\frac{1}{S} + \frac{8}{9}\cdot\frac{S}{S^2+9} + \frac{C}{S^2+9}$$

Taking Inverse Laplace transform, weget

$y = \frac{1}{9}\cdot 1 + \frac{8}{9}\cdot\cos 3t + \frac{c}{3}\sin 3t$

But $y\left(\frac{\pi}{2}\right) = -1$

$\Rightarrow -1 = \frac{1}{9} + \frac{8}{9}\cdot\cos\frac{3\pi}{2} + \frac{c}{3}\sin\frac{3\pi}{2}$

$$\Rightarrow C = \frac{-9-1-8\cos\frac{3\pi}{2}}{3\sin\frac{3\pi}{2}} = \frac{10}{3}$$

$\therefore y = \frac{1}{9} + \frac{8}{9}\cos 3t + \frac{10}{9}\sin 3t$

3. Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+9y=\sec 3x$.

Ans. See Q. No. 3.(a), 2005.

4. Examine the consistency of the following system of equations and solve, if possible :

$$x+y+z=1$$
$$2x+y+2z=2$$
$$3x+2y+3z=5.$$

Ans. See Q. No. 3(ii)(a), 2006

5. If $W=\{(x,y,z)\in R^3 : x+y+z=0\}$, show that W is a subspace of R^3, and find a basis of W.

Ans. $W=\{(x,y,z)\in R^3 : x+y+z=0\}$

Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in W \therefore \alpha_1+\alpha_2+\alpha_3=0$

and $\beta=(\beta_1,\beta_2,\beta_3)\in W \therefore \beta_1+\beta_2+\beta_3=0$

$a\alpha+b\beta=(a\alpha_1+b\beta_1, a\alpha_2+b\beta_2, a\alpha_3+b\beta_3)$

Now, $a\alpha_1+b\beta_1+a\alpha_2+b\beta_2+a\alpha_3+b\beta_3=a(\alpha_1+\alpha_2+\alpha_3)+b(\beta_1+\beta_2+\beta_3)$

$=a.0+b.0=0 \quad \therefore a\alpha+b\beta\in W \quad \therefore$ W is a subspace.

Let the basis is $a=(a_1,a_2,a_3) \quad \therefore a_1+a_2+a_3=0 \Rightarrow a_1=-(a_2+a_3)$

$\therefore a=(-a_2-a_3, a_2, a_3)=a_2(-1,1,0)+a_3(-1,0,1)$

$\therefore a=L\{(-1,1,0),(-1,0,1)\} \quad \therefore$ The basis is $L\{-1,1,0),(-1,0,1)\}$

$\therefore$ The dimension is 2.

6. Examine whether the mapping $T: R^2 \rightarrow R^2$ defined by $T(x,y)=(2x-y,x)$ is linear,

Ans. Let $\alpha,\beta\in R^2$ and $\alpha=(\alpha_1,\alpha_2), \beta=(\beta_1,\beta_2)$

$T(\alpha)=T(\alpha_1,\alpha_2)=(2\alpha_1-\alpha_2,\alpha_1)$

$T(\beta)=T(\beta_1,\beta_2)=(2\beta_1-\beta_2,\beta_1)$

Now $T(a\alpha+b\beta)=T(a\alpha_1+b\beta_1, a\alpha_2+b\beta_2)=[2(a\alpha_1+b\beta_1)-(a\alpha_2+b\beta_2), a\alpha_1+b\beta_1]$

$=[a(2\alpha_1-\alpha_2)+b(2\beta_2-\beta_2), a\alpha_1+b\beta_1]$

$=a(2\alpha_1-\alpha_2,\alpha_1)+b(2\beta_1-\beta_2,\beta_1)=aT(\alpha)+bT(\beta)$

$\therefore$ T is linear mapping.

7. **Evaluate $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal rule, taking four equal sub–intervals.**

Ans. $\int_0^1 \frac{dx}{1+x}$

Here $f(x) = \frac{1}{1+x}$, $a = 0$, $b = 1$, $n = 4$

$\therefore h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

x	$f(x) = \frac{1}{1+x}$	
0		1
0.25		0.8
0.5		0.6666
0.75		0.5714
1.0	0.5	
sum	1.5	2.0380

We Know from Trapctiodal rule

$$I_T = \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) + f(x_3)\} + f(x_4)\right]$$

$$= \frac{0.25}{2}[1 + 2(0.8 + 0.6666 + 0.5714) + 0.5]$$

$$= \frac{1}{8}[1.5 + 4.076] = \frac{5.576}{8} = 0.695$$

GROUP–C
(Long Answer Type Questions)
Answer any *three* of the following questions.

8. (a) **Find Laplace transform of $f(t) = \sin t, 0 < t < \pi$**
$= 0, t > \pi$.

(b) **If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then verify that A satisfies its own characteristic equation.**

Hence find A^{-1}

(c) Show that (3, 1, –2), (2, 1, 4), { 1, –1, 2) form a basis of R^3.

Ans. (a) We know from Definition

$$L\{f(t)\} = \int_0^{\alpha} e^{-st} f(t)dt = \int_0^{\pi} e^{-st}\sin t dt = \int_0^{\pi} e^{-st}\frac{e^{it}-e^{-it}}{2i}dt$$

$$= \frac{1}{2i}\left[\frac{e^{(i-s)t}}{i-s} + \frac{e^{-(i+s)t}}{i+s}\right]_0^{\pi}$$

$$= \frac{1}{2i}\left[\frac{e^{(i-s)\pi}}{i-s} + \frac{e^{-(i+s)\pi}}{i+s} - \left(\frac{1}{i-s}+\frac{1}{i+s}\right)\right]$$

$$= \frac{1}{2i}\left(-e^{-s\pi}-1\right)\left(\frac{1}{i-s}+\frac{1}{i+s}\right) = \frac{1+e^{-s\pi}}{1+s^2}$$

Ans. (b) $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda+\lambda^2-1) = -\lambda^3+2\lambda-1$

$\therefore$ The Charecteristic equation is $A^3-2A+I=0$

Here $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, $A^3 \begin{pmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{pmatrix}$

Now $A^3 - 2A+I = \begin{pmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 4 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore$ **A satisfies his own characteristic equation.**

$A^3-2A+I=0$

$\Rightarrow 2A - A^3 = I$

$\Rightarrow A(2-A^2) = I$

$\Rightarrow A(2I-A^2)=I$

$$\therefore A^{-1} = 2I - A^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Ans. (c) $\begin{vmatrix} 3 & 1 & -2 \\ 2 & 1 & 4 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -2 \\ -1 & 0 & 6 \\ 4 & 0 & 0 \end{vmatrix} = 24$

$\therefore$ The three vectors are linearly independent.
The vectors belong to R^3
$\therefore$ The veetors form a basis.

9. (a) Solve : $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x.$

(b) Solve : $(D^2 + 4)\, y = x \sin^2 x$, **where,** $D = \frac{d}{dx}$.

(c) Solve : $x\frac{dy}{dx} + y = y^2 \log x$. ....(1)

Ans. (a) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

Let $x = e^t$ i.e. $t = \log x$.

Then $x^2 \frac{d^2y}{dx^2} = D(D-1)y$ where $D = \frac{d}{dt}$

$x\frac{dy}{dx} = Dy$

The given equation (I) becomes

$D(D-1)y - Dy - 3y = c^{2t} \cdot t$

$\Rightarrow (D^2 - 2D - 3)y = te^{2t}$

Let the trial solution is $y = e^{mt}$

$\therefore$ A.E. $m^2 - 2m - 3 = 0 \Rightarrow m = 3, -1$

$\therefore$ C.F. is $Ae^{3t} + Be^{-t} = Ax^3 + \frac{B}{x}$

$$RI = \frac{1}{D^2-2D-3}\cdot te^{2t} = \frac{1}{(D-3)(D+1)}te^{2t} = e^{2t}\frac{1}{(D+2-3)(D+2+1)^1}$$

$$= e^{2t}\frac{1}{(D-1)(D+3)}t = C^{2t}\frac{1}{D^2+2d-s}t$$

$$= \frac{e^{2t}}{-3}\left(1-\tfrac{2D+D^2}{3}\right)^{-1}t = -\frac{e^{2t}}{3}\left(1+\tfrac{2D+D^2}{3}+\ldots\right)t = -\frac{e^{2t}}{3}\left(t+\tfrac{2}{3}\right)$$

$$= \frac{-e^{2t}}{3}t-\tfrac{2}{9}e^{2t} = \frac{-x^2\log x}{3}-\frac{2}{9}x^2$$

$\therefore$ The general solation is $y = Ax^3+\frac{B}{x}-\frac{x^2\log x}{3}-\frac{2}{9}x^2$

Ans. (b) Let $y = e^{mx}$ be the trial solution

$\therefore$ A. E. $= m^2+4=0 \Rightarrow m = \pm 2i$ $\quad\therefore$ CF $= A\cos 2x + B\sin 2x$

$$P.I = \frac{1}{D^2+4}x\sin^2 x = \frac{1}{D^2+4}\cdot\frac{x(1-\cos 2x)}{2}$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}\cdot x-\frac{1}{D^2+4}x\cos 2x\right] = \tfrac{1}{2}(P-Q)$$

$$P = \frac{1}{D^2+4}x = \frac{1}{4}\left(1+\tfrac{D^2}{4}\right)^{-1}x = \frac{x}{4}$$

$$Q = \frac{1}{D^2+4}x\cos 2x = \left(x-\frac{1}{D^2+4}\cdot 2D\right)\frac{1}{D^2+4}\cos 2x$$

$$= \left(x-\frac{1}{D^2+4}\cdot 2D\right)x\frac{1}{2D}\cos 2x$$

$$= \left(x-\frac{1}{D^2+4}\cdot 2D\right)\frac{x\sin 2x}{4} \qquad \left[\because \tfrac{1}{D}\cos 2x = \int\cos 2x\,dx\right]$$

$$= \frac{x^2\sin 2x}{4}-\frac{1}{D^2+4}\cdot\frac{\sin 2x+2x\cos 2x}{2}$$

$$= \frac{x^2\sin 2x}{4}-\frac{1}{2}\cdot\frac{1}{D^2+4}\sin 2x-\frac{1}{D^2+4}x\cos 2x$$

$$= \frac{x^2\sin 2x}{4}-\frac{1}{2}\cdot x\cdot\frac{1}{2}\sin 2x-Q = \frac{x^2\sin 2x}{4}+\frac{x\cos 2x}{8}-Q$$

$$\Rightarrow 2Q = \frac{x^2\sin 2x}{4}+\frac{x\cos 2x}{8}$$

$$\Rightarrow Q = \frac{x^2 \sin 2x}{8} + \frac{x\cos 2x}{16}$$

$$\therefore PI = \frac{1}{2}(P - Q) = \frac{x}{8} - \frac{x^2 \sin 2x}{16} - \frac{x\cos 2x}{32}$$

$\therefore$ The general solution is $y = A\cos 2x + B\sin 2x + \frac{x}{8} - \frac{x^2 \sin 2x}{16} - \frac{x\cos 2x}{32}$

Ans. (c) $x\frac{dy}{dx} + y = y^2 \log x$

$$\Rightarrow \frac{1}{y^2}\cdot\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x} \quad \text{.................(1)}$$

Let $Z = \frac{1}{y} \Rightarrow -\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{dz}{dx}$

(1) becomes : $\frac{dz}{dx} - \frac{z}{x} = -\frac{\log}{x}$ ....(2)

I. F. $= e^{-\int \frac{dx}{x}} = \frac{1}{x}$

Multiplying both sides of (2) by $\frac{1}{x}$ and then integrating we get

$$\frac{z}{x} = -\int \frac{\log x}{x^2}dx = -\left[\log x \cdot \int \frac{1}{x^2}dx - \int\left\{\frac{1}{x}\int\frac{dx}{x^2}\right\}dx\right]$$

$$= -\left[-\frac{\log x}{x} + \int\frac{dx}{x^2}\right] = -\left[-\frac{\log x}{x} - \frac{1}{x}\right] + C$$

$$\Rightarrow \frac{1}{xy} = \frac{\log x + 1}{x} - C$$

$$\Rightarrow \frac{1}{y} = (1 + \log x) - cx$$

10. (a) Compute f(0–5) & f (0–9) from the following table

x	**0**	**1**	**2**	**3**
***f*(x)**	**1**	**2**	**11**	**34**

(b) Apply the convolution theorem to evaluate

$$L^{-1}\left(\frac{1}{(s^2+2s+5)^2}\right)$$

(c) Prove that :

$$\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} = 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a).$$

Ans. (a) The difference lable is

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	1	8	6
1	2	9	14	
2	11	23		
3	34			

For evaluating f (0.5) we use Newton's Forward interpolation formula.

Here $x = 0.5$, $x_0 = 0$, $h = 1$, $u = \frac{x - x_0}{h} = 0.5$

$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{6}\Delta^3 f(x_0)$$

$$\Rightarrow f(0.5) = 1 + 0.5\times 1 + \frac{0.5(0.5-1)}{2}\times 8 + \frac{0.5(0.5-1)(0.5-2)}{6}\times 6 = 0.875$$

For evaluating f (0.9)

Here $x = 0.9$, $x_0 = 1$, $h = 1$, $u = \frac{x - x_0}{h} = -0.1$

$$\therefore f(0.9) = 2 - 0.1\times 9 + \frac{(0.1+1)}{2}\times 14 = 1.870$$

Ans. (b) $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+2s+5}\cdot\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{(S+1)^2+2^2}\cdot\frac{1}{(S+1)^2+2^2}\right\}$

we know. $L^{-1}\left\{\frac{1}{S^2+2^2}\right\} = \frac{1}{2}\sin 2t \quad \therefore L^{-1}\left\{\frac{1}{(S+1)^2+2^2}\right\} = \frac{1}{2}e^{-t}\sin 2t$

and we know from convolution theorem : $L^{-1}\{f(s)g(s)\} = \int^{t} F(u)G(t-u)\,du$

Where $L^{-1}\{f(s)\}=F(t)$ and $L^{-1}\{g(s)\}=G(t)$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2+2^2(s+1)^2+2^2}\right\}=\int_0^t \frac{1}{2}e^{-\mu}\,\text{Sin}2\mu\cdot\frac{1}{2}e^{-(t-u)}\text{Sin}2(t-u)du$$

$$=\int_0^t \frac{e^{-t}2\sin(2t-2u)\sin 2u}{8}du$$

$$=\frac{e^{-t}}{8}\int_0^t\left[\cos(2t-4u)-\cos 2t\right]du=\frac{e^{-t}}{8}\left[\frac{\sin(2t-4u)}{-4}-u\cos 2t\right]_0^t$$

$$=\frac{e^{-t}}{8}\left[\frac{\sin 2t}{4}-t\cos 2t+\frac{\sin 2t}{4}\right]=\frac{e^{-t}}{8}\left[\frac{\sin 2t}{2}-t\cos 2t\right]$$

Ans. (c) We know $\begin{vmatrix}x^2 & x & 1\\ y^2 & y & 1\\ z^2 & z & 1\end{vmatrix}\begin{matrix}R_2 & -R_1\\ R_3 & -R_1\end{matrix}\begin{vmatrix}x^2 & x & 1\\ y^2-x^2 & y-x & 0\\ z^2-x^2 & z-x & 0\end{vmatrix}=(y-x)(z-x)\begin{vmatrix}x^2 & x & 1\\ y+x & 1 & 0\\ z+x & 1 & 0\end{vmatrix}$

$=(y-x)(z-x)(y+x-z-x)=(y-x)(z-x)(y-z)$
$=-(x-y)(y-z)(z-x)$

and $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$

$$\therefore \begin{vmatrix}x^2 & -2x & 1\\ y^2 & -2y & 1\\ z^2 & -2z & 1\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$$

$$\Rightarrow\begin{vmatrix}(x-a)^2 & (x-b)^2 & (x-c^2)\\ (y-a)^2 & (y-b)^2 & (y-c)^2\\ (z-a)^2 & (z-b)^2 & (z-c)^2\end{vmatrix}=2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$$

11. (a) Prove that : $\Delta\cdot\nabla=\Delta-\nabla$, where symbols have their usual meaning.
(b) Estimate the missing term in the following table :

x	0	1	2	3	4
f(x)	1	3	9	—	81

(c) Solve : $(x^2y-2xy^2)\,dx+(3x^2y-x^3)\,dy=0$.

Ans. (a) $\Delta\cdot\nabla[f(x)] = \Delta[\nabla f(x)] = \Delta\{f(x)-f(x-h)\}$

$= \Delta f(x) - \Delta f(x-h) = \Delta f(x) - [f(x)-f(x-h)] = \Delta f(x) - \nabla f(x)$

$= (\Delta - \nabla) f(x) \quad \therefore \ \Delta\cdot\nabla = \Delta - \nabla$

Ans. (b) Four Points of f (x) are known

$\therefore \ \Delta^4 f(x) = 0$

$\Rightarrow (E-1)^4 f(x) = 0$

$\Rightarrow E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4E f(x) + f(x) = 0$

$\Rightarrow f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x) = 0$

Here h = 1 putting h = 1 and x = 0 we get

$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$$\Rightarrow f(3) = \frac{f(4)+6f(2)-4f(1)+f(0)}{4} = 31$$

Ans. (c) Method I

$$\frac{dy}{dx} = \frac{2xy^2 - x^2y}{3x^2y - x^3} \quad \text{..........................(1)}$$

It is homogeneous differential equation.

Let $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

(1) becomes $v + x\dfrac{dv}{dx} = \dfrac{2v^2x^3 - vx^3}{3vx^3 - x^3} = \dfrac{2v^2 - v}{3v-1}$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v^2-v}{3v-1} - v = -\frac{v^2}{3v-1}$$

$$\Rightarrow \frac{3v-1}{v^2}dv + \frac{dx}{x} = 0 \ \Rightarrow \left(\frac{3}{v} - \frac{1}{v^2}\right)dv + \frac{dx}{x} = 0$$

Integrating we get,

$3 \log v + \frac{1}{v} + \log x = c.$

$$\Rightarrow \log\frac{y^3}{x^2} + \frac{x}{y} = c$$

Method II $\dfrac{\partial M}{\partial y} = x^2 - 4xy, \dfrac{\partial N}{\partial x} = 6xy - 3x^2$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\left(10xy - 4x^2\right) \neq 0$$

$\therefore$ It is not exact differential equation

Now $Mx + NY = x^2 y^2$

$\therefore$ integrating factor is $\frac{1}{x^2y^2}$

Multiplying both sides of the given equation we get.

$$\frac{x^2y-2xy^2}{x^2y^2}dx+\frac{3x^2y-x^3}{x^2y^2}dy=0 \Rightarrow \frac{ydx-xdy}{y^2}+\frac{3}{y}dy-\frac{2}{x}dx=0$$

Integrating, we get $\frac{x}{y}+\log\frac{y^3}{x^2}=C$

12. (a) Show that $\frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x)$

(b) Obtain a function whose 1st difference is $9x^2+11x+5$, where $h = 1$.

(c) Solve the differential equation by Laplace transform :

$$\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=e^t \sin t\right), y(0)=0, y'(0)=1$$

Ans. (a) We know $J_n(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\Gamma(n+r+1)}$

$$\Rightarrow x^nJ_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{x^{2n+2r}}{2^{n+2r}\ r!\Gamma(n+r+1)}$$

$$\therefore \frac{d}{dx}\left[x^nJ_n(x)\right]=\sum_{r=0}^{\infty}(-1)^r\frac{2n+2r}{2^n+2\gamma}\cdot\frac{x^{2n+2r-1}}{r!\Gamma(n+r+1)}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{(n+r)}{r!(n+r)\Gamma(n+r)}\cdot\left(\frac{x}{2}\right)^{n+2r-1}\cdot x$$

$$=x^n\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{r!\Gamma(n-1+r+1)}=x^nJ_{n-1}(x)$$

Ans. (b) We first express $\Delta f(x)$ in term of factorial natation

1	9	11	5
	0	9	
	9	20	
	0		
	9		

$\therefore \Delta f(x) = 5\cdot x^{(0)} + 20\cdot x^{(1)} + 9\cdot x^{(2)}$

$\therefore \quad f(x) = 5x^{(1)} + \frac{20}{2}x^{(2)} + \frac{9}{3}x^{(3)} + A$, A is constant.

$= \quad 5\cdot x + 10x(x-1) + 3x(x-1)(x-2) + A$

$= \quad 5x + 10x^2 - 10x + 3x^3 - 9x^2 + 6x + A$

$= \quad 3x^3 + x^2 + x + A$

Ans. (c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^x \sin x$

taking laplace transform both side, weget

$\Rightarrow s^2y(s) - sy(o) - y'(0) + 2sy(s) - 2y(0) + 5y(s) = \frac{1}{(s-1)^2 + 1^2}$

$\Rightarrow (s^2 + 2s + 5)y(s) - 1 = \frac{1}{s^2 - 2s + 2}$

$\Rightarrow y(s) = \frac{1}{(s^2 - 2s + 2)(s^2 + 2s + 5)} + \frac{1}{s^2 + 2s + 5} = \frac{s^2 - 2s + 36}{(s^2 - 2s + 2)(s^2 + 2s + 5)}$

$\Rightarrow y(s) = \frac{1}{65}\left[\frac{4(S+1)}{(S+1)^2 + 2^2} + \frac{66}{(S+1)^2 + 2^2} - \frac{4(S-1)}{(S-1)^2 + 1} + \frac{7}{(S-1)^2 + 1}\right]$

Toking inverse Laplace transformation, we get

$\Rightarrow \quad y = \frac{1}{65}\left[4e^{-x}\cos 2x + 33e^{-x}\sin 2x - 4e^x \cos x + 7e^x \sin x\right]$

$= \frac{e^{-x}}{65}(4\cos 2x + 33\sin 2x) + \frac{e^{x}}{65}(7\sin x - 4\cos x)$

13. (a) What is meant by linear independence of a set of n–vectors ? Show that the vectors (1, 2, 1), (–1, 1, 0) and (5, –1, 2) are linearly independent.

(b) Show that $W = \{(x, y, z) \in R^3 : 2x - y + 3z = 0\}$ is a subspace of R^3. Find a basis of *W*. What is its dimension ?

(c) Prove that $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$

Ans. (a) Linear Independence of a set of n vectors :–

Let F be a field. The n vectors $v_1, v_2, \ldots v_n$ is said to be linearly independent if every relation of the form

$c_1v_1 + c_2v_2 + \ldots + c_n v_n = 0$, $c_i \in F$, $1 \le i \le n$ implies $c_i = 0$, for each $1 \le i \le n$.

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 5 & -1 & 2 \end{vmatrix} = 2 \neq 0 \quad \therefore \text{ The given vectors are linearly independent.}$$

Ans. (b) See Q. No. 3.(a), 2002.

Ans. (c) See Q. No. 2.(a), 2004.

14. (a) Find the rank of matrix $\begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$

(b) Solve by Cramer's rule :

$2x - y = 3$
$3y - 2z = 5$
$-2z + x = 4$

(c) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$$

Ans. (a) See 2002, 2b

Ans. (b) Here $\Delta = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & -2 \end{vmatrix} = 2(-6) + 1(2) = -10$

$$\Delta_1 = \begin{vmatrix} 3 & -1 & 0 \\ 5 & 3 & -2 \\ 4 & 0 & -2 \end{vmatrix} = 3(-6) + 1(-10+8) = -20$$

$$\Delta_2 = \begin{vmatrix} 2 & 3 & 0 \\ 0 & 5 & -2 \\ 1 & 4 & -2 \end{vmatrix} = 2(-10+8) - 3(2) = -10$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 3 & 5 \\ 1 & 0 & 4 \end{vmatrix} = 2(12) + 1(-5-9) = 10$$

$\therefore\ x = \frac{\Delta_1}{\Delta} = \frac{-20}{-10} = 2$

$y = \frac{\Delta_2}{\Delta} = \frac{-10}{-10} = 1$

$z = \frac{\Delta 3}{\Delta} = \frac{10}{-10} = -1$

$\therefore$ The solution is $x = z, y = 1, z = -1$

Ans. (c) The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & -2-\lambda & 4 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$$

$\Rightarrow\ \lambda^2(5-\lambda) = 0$

$\Rightarrow\ \lambda = 0,0,5$

$\therefore$ The eigen values are 0, 0, 5.

When $\lambda = 0$, the eigen vector is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\therefore \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 - x_2 + 2x_3 = 0$

$2x_1 - 2x_2 + 4x_3 = 0$

$3x_1 - 3x_2 + 6x_3 = 0$

$\because$ The determinant of the coefficient matrix is equal to zero.

$\therefore$ The system has infinite solutions.

$x_1 - x_2 + 2x_3 = 0$

$\Rightarrow x_2 = x_1 + 2x_3$

$\therefore$ The eigen vector is $\begin{pmatrix} x_1 \\ x_1 + 2x_3 \\ x_3 \end{pmatrix}$

When $\lambda = 5$

$$\begin{pmatrix} -4 & -1 & 2 \\ 2 & -7 & 4 \\ 3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad -4y_1 - y_2 + 2y_3 = 0$ ....(1)

$2y_1 - 7y_2 + 4y_3 = 0$ ......(2)

$3y_1 - 3y_2 + y_3 = 0$ ....(3)

(3) × 2 – (1) gives $2y_1 - y_2 = 0$

(3) 2 gives $2y_1 - y_2 = 0$

$\therefore \; y_2 = 2y_1$

From (3)

$y_3 = 3y_2 - 3y_1$

$= 6y_1 - 3y_1$

$= 3y_1$

$\therefore$ The eigen veefor is $\begin{pmatrix} y_1 \\ 2y_1 \\ 3y_1 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$