ENGINEERING MATHEMATICS—2007

Time : 3 Hours *Full Marks : 70*

Group–A

[Multiple Choice Type Questions]

1. **Choose the correct alternatives for the following :**

$10 \times 1 = 10$

(i) Laplace transform of the function cos (at) is

(a) $\frac{s}{s^2 - a^2}$ (b) $\frac{a}{s^2 + a^2}$ (c) $\frac{s}{s^2 + a^2}$ (d) $\frac{1}{s^2 - a^2}$

Ans. –e

(ii) The rank of the matrix $A = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$ is

(a) 0 (b) 1 (c) 3 (d) 2

Ans. $A = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

There are two non zero rows

$\therefore$ Rank is 2.

(iii) The order and degree of the differential equation $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = y$ are

(a) 2,2 (b) 2,1 (c) 1, 2 (d) 1, 1

Ans. (b)

(iv) Value of the determinant $\begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix}$ is

(a) 0 (b) abc (c) –abc (d) 2 abc

Ans. (a)

(v) Integrating factor of $\frac{dy}{dx} + y = 1$ is

(a) e^x (b) x^2 (c) x (d) 2

Ans. Integrating factor is $e^{\int dx} = e^x$

(vi) The operator equivalent to shift operator E is

(a) $1 + \Delta$ (b) $(I + \Delta)^{-1}$ (c) $I - \Delta$ (d) $I - \Delta^2$

Ans. (a)

(vii) The number of significant digits in 3.0044 is

(a) 5 (b) 2 (c) 3 (d) 4

Ans. (a)

(viii) If α, β are the roots of the equation $x^2 - 3x + 2 = 0$, then $\begin{vmatrix} 0 & \alpha & \beta \\ \beta & 0 & 0 \\ 1 & -\alpha & \alpha \end{vmatrix}$ is

(a) 6 (b) $\frac{3}{2}$ (c) -6 (d) 3

Ans. α, β are the roots of the equation $x^2 - 3x + 2 = 0$

$\therefore \alpha = 1, \beta = 2$

The determinant is $\begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} = -2(1+2) = -6$

(ix) The sum of the eigen values of $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix}$ is

(a) 5 (b) 2 (c) 1 (d) 6

Ans. (a)

(x) $(\Delta - \nabla)\, x^2$ is equal to

(a) h^2 (b) $-2h^2$

(c) $2h^2$ (d) None of these

Ans. $(\Delta - \nabla)\, x^2 = \Delta x^2 - \nabla x^2$

$$= (x+h)^2 - x^2 \{x^2 - (x-h)^2\}$$
$$= (x+h)^2 - x^2 - x^2 + (x-h)^2$$
$$= x^2 + 2hx + h^2 - 2x^2 + x^2 - 2hx + h^2$$
$$= 2h^2$$

(xi) If a linear transformation $T : IR^2 \to IR^2$ be defined by

$T(x_1, x_2) = (x_1 + x_2, 0)$ then ker (T) is

(a) $\{(1, -1)\}$ (b) $\{(1, 0)\}$ (c) $\{(0, 0)\}$ (d) $\{(1, 0), (0, 1)\}$

Ans. Let ker (T) is (x, y)

$\therefore T(x, y) = \theta$

$\Rightarrow (x + y, 0) = (0, 0)$

$\Rightarrow x + y = 0$

$\Rightarrow x = -y = k$

$\Rightarrow x = k, y = -k$

$\therefore \ker(T) = (x, y) = k(1, -1) \qquad \therefore \text{Ker}(T)\ \{(1, -1)\}$

(xii) The value of $\begin{vmatrix} 2000 & 2001 & 2002 \\ 2003 & 2004 & 2005 \\ 2006 & 2007 & 2008 \end{vmatrix}$ is

(a) 2000 (b) 0 (c) 45 (d) None of these.

Ans. $\begin{vmatrix} 2000 & 2001 & 2002 \\ 2003 & 2004 & 2005 \\ 2006 & 2007 & 2008 \end{vmatrix} \xrightarrow{C_2 - C_1} \begin{vmatrix} 2000 & 1 & 1 \\ 2003 & 1 & 1 \\ 2004 & 1 & 1 \end{vmatrix} = 0$

(xiii) The value of K for which the vectors (1, 2, 1), (K, 1, 1) and (0, 1, 1) are linearly dependent

(a) 1 (b) 2 (c) 0 (d) 3

Ans. $\begin{vmatrix} 1 & 2 & 1 \\ k & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ k & 0 & 6 \\ 0 & 1 & 1 \end{vmatrix} = -k(2-1) = -k \quad \therefore k = 0$

Group–B

[Short Answer Type Questions]

Answer *any three* questions. $3 \times 5 = 15$

2. Apply convolution theorem to find the inverse Laplace transform of $\dfrac{s}{(s^2+9)^2}$

Ans. We have $L^{-1}\left\{\dfrac{s}{s^2+9}\right\} = \cos 3t$ and $L^{-1}\left\{\dfrac{1}{s^2+9}\right\} = \dfrac{\sin 3t}{3}$

Now using convolution theorem, we get

$$L^{-1}\left\{\frac{1}{(s^2+9)}\right\} = \int_0^t \cos 3u \frac{\sin 3(t-u)}{3}\, du = \frac{1}{3}\int_0^t (\cos 3u)(\sin 3t \cos 3u - \cos 3t \sin 3u)\, du$$

$$= \frac{1}{3} \sin 3t \int_0^t \cos^2 3u\, du - \frac{1}{3} \cos 3t \int_0^t \cos 3u \sin 3u\, du$$

$$= \frac{1}{3} \sin 3t \int_0^t \frac{1+\cos 6u}{2} du - \frac{1}{3} \cos 3t \int_0^t \frac{\sin 3u \, d(\sin 3u)}{3}$$

$$= \frac{1}{3} \sin 3t \left[\frac{u}{2} + \frac{\sin 6u}{12}\right]_0^t - \frac{1}{9} \cos 3t \left[\frac{\sin^2 3u}{2}\right]_0^t$$

$$= \frac{1}{3} \sin 3t \left[\frac{t}{2} + \frac{\sin 6u}{12}\right]_0^t - \frac{1}{9} \cos 3t \left[\frac{\sin^2 3u}{2}\right]_0^t$$

$$= \frac{1}{3} \sin 3t \left[\frac{t}{2} + \frac{\sin 6t}{12}\right] - \frac{1}{18} \cos 3t \sin^2 3t$$

$$= \frac{t \sin 3t}{6} + \frac{\sin 3t \sin 6t}{36} = \frac{1}{18} \cos 3t \sin^2 3t$$

$$= \frac{t \sin 3t}{6} + \frac{\cos 3t \sin^2 3t}{18} - \frac{\cos 3t \sin^2 3t}{18} = \frac{t \sin 3t}{6}$$

3. Prove that $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$.

Ans. $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$ $R_1 = R_1 - R_2 - R_3$

$$= 2b^2c^2 \begin{vmatrix} 0 & -c^2 & -b^2 \\ 1 & \frac{c^2+a^2}{b^2} & 1 \\ 1 & 1 & \frac{a^2+b^2}{c^2} \end{vmatrix} = 2b^2c^2 \begin{vmatrix} 0 & -c^2 & -b^2 \\ 0 & \frac{c^2+a^2}{b^2} - 1 & 1 - \frac{a^2+b^2}{c^2} \\ 1 & 1 & \frac{a^2+b^2}{c^2} \end{vmatrix} \quad R_2 = R_2 - R_3$$

$$= 2b^2c^2 \left\{-c^2\left(1 - \frac{a^2+b^2}{c^2}\right) + b^2\left(\frac{a^2+c^2}{b^2} - 1\right)\right\} = 2b^2c^2\{-c^2 + a^2 + b^2 + a^2 + c^2 - b^2\}$$

$$= 4a^2b^2c^2$$

4. Solve the differential equation by Laplace transform :

$$\frac{d^2x}{dt^2}+4x=\sin\ 3t,\ x(0),\ x'(0)=0$$

Ans. $\frac{d^2x}{dt^2}+4x=\sin\ 3t$

$$\Rightarrow L\left\{\frac{d^2x}{dt^2}\right\}+L\{4x\}=L\{\sin\ 3t\}$$

$$\Rightarrow s^2L(x)-sx(0)-x'(0)+4L(x)=\frac{3}{s^2+9}$$

$$\Rightarrow L(x)\ (s^2+4)=\frac{3}{s^2+9}+sx(0)+x'(0)$$

$$\Rightarrow L(x)\ (s^2+4)=\frac{3}{s^2+9}$$

$$\Rightarrow L(x)=\frac{3}{(s^2+9)(s^2+4)}=\frac{3}{5}\left(\frac{1}{s^2+4}-\frac{1}{s^2+9}\right)=\frac{3}{10}\cdot\frac{2}{s^2+4}-\frac{1}{5}\cdot\frac{3}{s^2+9}$$

$$\Rightarrow x=\frac{3}{10}\ \sin 2t-\frac{1}{5}\ \sin 3t$$

5. Solve : $(x^2+y^2+2x)\ dx+xy\ dy=0$

Ans. $(x^2+y^2+2x)dx+xy\ dy=0$

$\Rightarrow (x^3+xy^2+2x^2)\ dx+x^2y\ dy=0$

$\Rightarrow (x^3+2x^2)dx+xy^2\ dx+x^2y\ dy=0$

$\Rightarrow (x^3+2x^2)\ dx+d\ (x^2\ y^2)=0$

Integrating $\frac{x^4}{4}+\frac{2x^3}{3}+\frac{x^2y^2}{2}=C$

6. Show that $W=\{(x, y, z)\in R^3, 2x-y+3z=0)$ is a subspace of R^3. Find a basis of W. What its dimension?

Ans. Let $\alpha=(\alpha_1, \alpha_2, \alpha_3)$ and $\beta=(\beta_1, \beta_2, \beta_3)$, $\alpha,\beta\in w$

$\therefore\ 2\alpha_1-\alpha_2+3\alpha_3=0$ and $2\beta_1-\beta_2+3\beta_3=0.$

$a\alpha+b\beta=a(\alpha_1, \alpha_2, \alpha_3)+b(\beta_1, \beta_2, \beta_3)=(a\alpha_1+b\beta_1, a\alpha_2+b\beta_2, a\alpha_3+b\beta_3)$

Now $2(a\alpha_1+b\beta_1)-(a\alpha_2+b\beta_2)+3(a\alpha_3+b\beta_3)=a(2\alpha_1-\alpha_2+3\alpha_3)+b(2\beta_1-\beta_2+3\beta_3)$

$=a\ .\ 0+b\ .\ 0=0$

$\therefore a\alpha + b\beta \in w \quad \therefore$ w is a subspace of R^3.

$\alpha = (\alpha_1, \alpha_2, \alpha_3) \in w \qquad \therefore 2\alpha_1 - \alpha_2 + 3\alpha_3 = 0$

$\Rightarrow \alpha_2 = 2\alpha_1 + 3\alpha_3$

$\therefore \alpha = (\alpha_1, 2\alpha_1 + 3\alpha_3, \alpha_3) = \alpha_1 (1, 2, 0) + \alpha_3 (0, 3, 1) \quad \therefore \alpha = L\{(1, 2, 0), (0, 3, 1)\}$

$\therefore$ Basis of W is $\{(1, 2, 0), (0, 3, 1)\}$ and dimension is 2.

7. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then verify that A satisfies its own characteristic equation. Hence find A^{-1}.

Ans. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Characteristic equation is $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)\{(-1 - \lambda)(-\lambda) - 1\} = 0$

$\Rightarrow (1 - \lambda)(\lambda^2 + \lambda - 1) = 0$

$\Rightarrow \lambda^3 - 2\lambda + 1 = 0$

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

Now $A^3 - 2A + I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ A satisfies its own characteristic equation.

$A^3 - 2A + I = 0$

$\Rightarrow 2A - A^3 = I$

$\Rightarrow A(2I - A^2) = I$

$\therefore A^{-1} = 2I - A^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Group–C

(Long Answer Type Questions)

Answer *any three* questions. $3 \times 15 = 45$

8. (a) **Evaluate** $\left(\frac{\Delta^2}{E}\right)x^3$. 5

(b) **Find the missing data in the following table :** 5

x	–2	–1	0	1	2
f(x)	6	0	....	0	6

(c) **Show that (3, 1, –2), (2, 1, 4) and (1, –1, 2) form a basis of R^3.** 5

Ans. (a) $\left(\frac{\Delta^2}{E}\right)x^3 = \left\{\frac{(E-1)^2}{E}\right\}x^3 = \left(E - 2 + E^{-1}\right)x^3$

$$= (x+h)^3 - 2x^3 + (x-h)^3 = 6xh^2$$

Ans. (b) There are known four points.

∴ The polynomial obtained from the given points is of degree 3

∴ Fourth difference is equal to zero. ∴ $\Delta^4 f(x) = 0$

$\Rightarrow (E-1)^4 f(x) = 0$

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) f(x) = 0$

$\Rightarrow f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$

Putting $x = -2$

$f(2) - 4f(1) + 6f(0) - 4f(-1) + f(-2) = 0$

$\Rightarrow 6 - 0 + 6f(0) - 0 + 6 = 0$

$\Rightarrow f(0) = -2,\ f(x) = 2x^2 - 2.$

Ans. (c) $\begin{vmatrix} 3 & 1 & -2 \\ 2 & 1 & 4 \\ 1 & -1 & 2 \end{vmatrix} = 24 \neq 0$ ∴ The vectors are linearly independent.

Since R^3 is a vector space of dimension 3 and there are 3 linearly independent vectors of R^3. Therefore the vectors form a basis.

9. (a) **Prove that** $\int_0^{+\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$, **by Laplace transform.** 5

(b) **Apply convolution theorem to prove that** $\int_0^t \sin u \cos(t-u) dt = \frac{t}{2}\sin t$. 5

(c) **Solve :** $\frac{dy}{dx} - \frac{\tan y}{(1+x)} = (1+x)e^x . \sec y$. 5

Ans. (a) Let $F(t) = \sin t \Rightarrow f(s) = \dfrac{1}{s^2+1}$

We know $\displaystyle\int_0^{\alpha} \frac{F(t)}{t}\, dt = \int_0^{\alpha} f(u)\, du$

$$\Rightarrow \int_0^{\alpha} \frac{\sin t}{t}\, dt = \int_0^{\alpha} \frac{1}{u^2+1}\, du = \left[\tan^{-1} u\right]_0^{\alpha} = \frac{\pi}{2}$$

Ans. (b) Let $F(t) \displaystyle\int_0^{t} \sin u \cos (t-u)\, dt$

Using convolution theorem, we get

$$L\{F(t)\} = L\{\sin t\}\, L\{\cos t\}$$

$$= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} = \frac{s}{\left(s^2+1\right)^2} = -\frac{1}{2} \frac{2s}{\left(s^2+1\right)^2}$$

$$= -\frac{1}{2} \frac{d}{ds}\left(\frac{1}{1+s^2}\right) = \frac{1}{2} (-1)^1 \frac{d}{ds}\left(\frac{1}{1+s^2}\right)$$

$$\Rightarrow F(t) = \frac{1}{2} L^{-1}\left\{(-1)^1 \frac{d}{ds}\left(\frac{1}{1+s^2}\right)\right\} = \frac{1}{2} t^1 \cdot \sin t = \frac{t \sin t}{2}$$

Ans. (c) $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)\, e^x \sec y$

$$\Rightarrow \cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)\, e^x \;\; \ldots\ldots. (1)$$

Let $V = \sin y \;\; \Rightarrow \dfrac{dv}{dx} = \cos y \dfrac{dy}{dx}$

(1) becomes $\dfrac{dv}{dx} - \dfrac{v}{1+x} = (1+x)\, e^x \;\; \ldots\ldots (2)$

$$\frac{1}{1+x} \frac{dv}{dx} - \frac{2}{(1+x)^2} = e^x$$

Integrating $\dfrac{v}{1+x} = \displaystyle\int e^x\, dx + c \qquad \Rightarrow \sin y = (1+x)\left(e^x + c\right)$

10. (a) Solve by Cramer's rule : 5

$x + y + z = 7$
$x + 2y + 3z = 15$
$x - y + z = 3$

(b) **Find general solution of $p = \cos(y - px)$, where $p = \frac{dy}{dx}$.** 5

(c) **Solve :** $\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y(\log y)^2}{x^2}$. 5

Ans. (a)
$$x + y + z = 7$$
$$x + 2y + 3z = 15$$
$$x - y + z = 3$$

Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 4$

$$\Delta_1 = \begin{vmatrix} 7 & 1 & 1 \\ 15 & 2 & 3 \\ 3 & -1 & 1 \end{vmatrix} = 8$$

$$\Delta_2 = \begin{vmatrix} 1 & 7 & 1 \\ 1 & 15 & 3 \\ 1 & 3 & 1 \end{vmatrix} = 8$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 7 \\ 1 & 2 & 15 \\ 1 & -1 & 3 \end{vmatrix} = 12$$

$\therefore\ x = \frac{\Delta_1}{\Delta} = 2,\ y = \frac{\Delta 2}{\Delta} = 2,\ z = \frac{\Delta 3}{\Delta} = 3 \qquad x = 2, y = 2\ z = 3$

Ans. (b) $p = \cos(y - px) \Rightarrow y - px = \cos^{-1} p$ ...... (1)

Diff. w.r.t. to x.

$$p - p - x\frac{dp}{dx} = \frac{-1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

$$\Rightarrow \left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \Rightarrow p = c$$

$\therefore$ The general solution is, from (1)

$$y = cx + \cos^{-1} c$$

$$x - \frac{1}{\sqrt{1-p^2}} = 0 \Rightarrow x = \frac{1}{\sqrt{1-p^2}} \Rightarrow p = \frac{\sqrt{x^2-1}}{x}$$

$\therefore$ The singular solution is $y = \sqrt{x^2-1} + \cos^{-1}\frac{\sqrt{x^2-1}}{x}$

Ans. (c). $\frac{dy}{dx}+\frac{y\log y}{x}=\frac{y(\log y)^2}{x^2}$ .... (1)

$\Rightarrow \frac{1}{y}\frac{dy}{dx}+\frac{\log y}{x}=\frac{(\log y)^2}{x^2}$ ......(2)

Let $V=\log y \Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{dv}{dx}$

From (2) $\frac{dv}{dx}+\frac{v}{x}=\frac{v^2}{x^2} \Rightarrow \frac{1}{v^2}\frac{dv}{dx}+\frac{1}{vx}=\frac{1}{x^2}$ ...... (3)

Let $t=\frac{1}{v} \Rightarrow \frac{1}{v^2}\frac{dv}{dx}=-\frac{dt}{dx}$

From (3)

$\frac{-dt}{dx}+\frac{t}{x}=\frac{1}{x^2}, \Rightarrow \frac{dt}{dx}-\frac{t}{x}=\frac{-1}{x^2} \Rightarrow \frac{1}{x}\frac{dt}{dx}-\frac{t}{x^2}=-\frac{1}{x^3}$

Integrating, we get $\frac{t}{x}=\frac{1}{2x^2}+c$

$\Rightarrow \frac{1}{\sqrt{x}}=\frac{1}{2x^2}+c \Rightarrow \frac{1}{x\log y}=\frac{1}{2x^2}+c \Rightarrow 2x=\log y\left(1+2cx^2\right)$

11. (a) Solve $(D^2-2D)y=e^x \sin x$, where $D\equiv\frac{d}{dx}$. 5

(b) Solve : $\frac{dx}{dt}-7x+y=0$ and $\frac{dy}{dt}-2x-5y=0$ 5

(c) Solve by Gauss-elimination method : 5

$2x+2y+z=12$

$3x+2y+2z=8$

$5x+10y-8z=10$

Ans. (a) $(D^2-2D)y=e^x\sin x \Rightarrow D(D-2)y=e^x\sin x.$

To find C.F.,

The A.E. $m(m-2)=0$ $m=0, 2$ $\quad\therefore CF=C_1+C_2e^{2x}$

To find P.F. $PI=\frac{1}{D(D-2)}e^x\sin x. = e^x\frac{1}{(D+1)(D-1)}\sin x. = e^x\frac{1}{D^2-1}\sin x$

$=-\frac{e^x\sin x}{2}$

$\therefore$ The solution is $y=C_1+C_2e^{2x}-\frac{e^x\sin x}{2}$

Ans. (b) $(D-7)x + y = 0$..............(1)

$(D-5)y - 2x = 0$ ....(2)

(1) × 2 + (2) × (D – 7) gives.

$(D^2 - 12D + 37)y = 0 \qquad \therefore y = e^{6t}(c_1 \cos t + c_2 \sin t)$

Putting the value of y in (2) we get, $e^{6t}[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t] - 2x = 0$

$\Rightarrow x = e^{6t}(k_1 \cos t + k_2 \sin t)$

where $k_1 = \dfrac{c_1 + c_2}{2}$, $k_2 = \dfrac{c_2 - c_1}{2}$

∴ The solution is

$x = e^{6t}(k_1 \cos t + k_2 \sin t)$

$y = e^{6t}(c_1 \cos t + c_2 \sin t)$

Ans. (c) The augmented matrix is

$$A = \begin{bmatrix} 2 & 2 & 1 & 12 \\ 3 & 2 & 2 & 8 \\ 5 & 10 & -8 & 10 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 2 & 2 & 1 & 12 \\ 1 & 0 & 1 & -4 \\ 0 & 6 & -11 & -10 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 0 & 2 & -1 & 20 \\ 1 & 0 & 1 & -4 \\ 0 & 6 & -11 & -10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 2 & -1 & 20 \\ 0 & 6 & -11 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 2 & -1 & 20 \\ 0 & 0 & -8 & -70 \end{bmatrix}$$

The system of linear equation of the matrix are

$-8z = -70 \Rightarrow z = 35/4$

$2y - z = 20 \Rightarrow y = 115/8$

$x + z = -4 \Rightarrow x = -51/4 \qquad \therefore$ The solution is $x = -\dfrac{51}{4}$, $y = \dfrac{115}{8}$, $z = \dfrac{35}{4}$

12. (a) Use Lagrange's interpolation formula to find f(x), where $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$ and $f(9) = 13104$. 5

(b) Apply appropriate interpolation formula to calculate f(2.1). Correct upto two significant figures from the following data : 5

x	0	2	4	6	8	10
f(x)	1	5	17	37	45	51

(c) Apply Simpson's 1/3rd rule of evaluate $\int_0^6 \frac{dx}{(1+x)^2}$ taking six equal intervals from [0, 6] and correct upto three decimal places. 5

Ans. (a) $f(x)=\frac{(x-1)(x-3)(x-5)(x-6)(x-9)}{(0-1)(0-3)(0-5)(0-6)(0-9)}\times(-18)$

$$+\frac{(x-0)(x-1)(x-3)(x-6)(x-9)}{(5-0)(5-1)(5-3)(5-6)(5-9)}\times(-248)+\frac{(x-0)(x-1)(x-3)(x-5)(x-6)}{(9-0)(9-1)(9-3)(9-5)(9-6)}\times(13104)$$

$$=(x-1)(x-3)(x-6)\left[\frac{(x-5)(x-9)}{45}-\frac{x(x-9)\times 31}{20}+\frac{x(x-5)91}{36}\right]$$

$$=\left(x^2-10x^2-27k-18\right)\times\left(\frac{4x^2-56x+180-279x^2+2511x+455x^2-2275x}{180}\right)$$

$$=\left(x^3-10x^2+27x-18\right)\left(\frac{180x^2+180x+180}{180}\right)$$

$$=(x^3-10x^2+27x-18)(x^2+x+1)=x^5-9x^4+18x^3-x^2+9x-18$$

Ans. (b) The difference table is

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	1	4	8	0	–20	–10
2	5	12	8	–20	–30	
4	17	20	– 12	10		
6	37	8	–2			
8	45	6				
10	51					

Here $h=2$, $x=2.1$, $x_0=2$, $u=\frac{x-x_0}{h}=\frac{2\cdot 1-2}{2}=0\cdot 05$

Applying Newton's forward interpolation formula.

$$f(x)=f(x_0)+u\,\Delta f(x_0)+\frac{u(u-1)}{2!}\Delta^2 y(x_0)+\frac{u(u-1)(u-2)}{3!}+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(x_0)$$

$$=5+0\cdot 05\times 12+\frac{(0\cdot 05)(0\cdot 05-1)}{2}\times 8$$

$$+\frac{(0\cdot 05)(0\cdot 05-1)(0\cdot 05-2)}{6}\times(-20)+\frac{(0\cdot 05)(0\cdot 01-1)(0\cdot 05-2)(0\cdot 05-3)}{24}\times(-30)$$

$= 5.442804688$

$= 5.4$

Ans. (c) Here $x_0=0$, $x_6=0$, $h=\frac{x_6-x_0}{6}=\frac{6-0}{6}=1$ and $f(x)=\frac{1}{(1+x)^2}$

x	f(x)
$x_0 = 0$	$y_0 = 1$
$x_1 = 1$	$y_1 = \frac{1}{4}$
$x_2 = 2$	$y_2 = \frac{1}{9}$
$x_3 = 3$	$y_3 = \frac{1}{16}$
$x_4 = 4$	$y_4 = \frac{1}{25}$
$x_5 = 5$	$y_5 = \frac{1}{36}$
$x_6 = 6$	$y_6 = \frac{1}{49}$

We know from Simpson's 1/3rd rule

$$I = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6\right]$$

$$= \frac{1}{3}\left[1 + 4\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36}\right) + 2\left(\frac{1}{9} + \frac{1}{25}\right) + \frac{1}{49}\right] = 0.895$$

13. (a) Apply the method variation of parameter to solve the equation $\frac{d^2y}{dx^2} + y = \sec^3 x \cdot \tan x$. 6

(b) Expand by Laplace's method 6

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + cd)^2$$

(c) Given that L{sint/t} = $\tan^{-1}(1/s)$, find L{sin(at)/t}. 3

Ans. (a) $\frac{d^2y}{dx^2} + y = \sec^3 x \tan x$

To find C.F., the A.E. is

$m^2 + 1 = 0 \Rightarrow m = \pm i \qquad \therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

$$\therefore W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$\therefore$ P.I. $= \sin x \int \cos x \,.\, \sec^3 x \tan x \, dx - \cos x \int \sin x \sec^3 x \tan x \, dx$

$$= \sin x \int \sec^2 x \tan x \, dx - \cos x \int \sec^2 x \tan^2 x \, dx$$

$$= \sin x \int \tan x \, d(\tan x) - \cos x \int \tan^2 x \, d(\tan x)$$

$$= \sin x \frac{\tan^2 x}{2} - \cos x \frac{\tan^3 x}{3} = \tan^2 x \left(\frac{\sin x}{2} - \frac{\sin x}{3}\right)^3 = \frac{\sin x \tan^2 x}{6}$$

$\therefore$ The solution is $y = c_1 \cos x + c_2 \sin x + \dfrac{\sin x \tan^2 x}{6}$

Ans. (b) $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}\begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix} - \begin{vmatrix} 0 & b \\ -a & d \end{vmatrix}\begin{vmatrix} -d & f \\ -e & 0 \end{vmatrix}$

$$+\begin{vmatrix} 0 & c \\ -a & e \end{vmatrix}\begin{vmatrix} -d & 0 \\ -e & -f \end{vmatrix} + \begin{vmatrix} a & b \\ 0 & d \end{vmatrix}\begin{vmatrix} -b & f \\ -c & 0 \end{vmatrix} - \begin{vmatrix} a & c \\ 0 & e \end{vmatrix}\begin{vmatrix} -b & 0 \\ -c & -f \end{vmatrix} + \begin{vmatrix} b & c \\ d & e \end{vmatrix}\begin{vmatrix} -b & -d \\ -c & -e \end{vmatrix}$$

$$= a^2f^2 - abef + acdf + acdf - abef + (be - cd)(be - cd)$$
$$= a^2f^2 - 2abef + 2acdf + b^2e^2 + c^2d^2 - 2bced$$
$$= (af - be + cd)^2$$

Ans. (c) We know, if $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$

Let $f(t) = \dfrac{\sin t}{t}$

$L\{f(t)\} = \bar{f}(s) = \tan^{-1}\dfrac{1}{s}$ $\quad\therefore L\{f(at)\} = L\left\{\dfrac{\sin at}{at}\right\} = \dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right) = \dfrac{1}{a}\tan^{-1}\dfrac{1}{s/a}$

$\Rightarrow \dfrac{1}{a} L\left\{\dfrac{\sin at}{t}\right\} = \dfrac{1}{a}\tan^{-1}\dfrac{a}{s}$ $\qquad \Rightarrow L\left\{\dfrac{\sin at}{t}\right\} = \tan^{-1}\dfrac{a}{s}$

14. (a) Assuming the orthogonal properties of Legendre function, prove that

$$\int_{-1}^{+1}\left[P_n(x)\right]^2 dx = \frac{2}{2n+1}$$ 5

(b) Show that $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$. 5

(c) State Cayley-Hamilton theorem and show that the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$ satisfies the above theorem. 5

Ans. (a) We have from Rodrigue's formula

$$p_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}\left(x^2-1\right)^n = \frac{1}{n!2^n}D^n\left(x^2-1\right)^n,\ D \equiv \frac{d}{dx} \quad ....(1)$$

The orthogonal property of Legendre polynomialsis

$$\int_{-1}^{1} p_m(x)\, p_n(x)dx = 0,\ (m \neq n) \quad ......\ (2)$$

From equn. (1) we have, $n!2^n\ p_n(x) = D^n\left(x^2-1\right)^n$ .....(3)

When m = n, then from (2) & (3), we have

$$\left(n!2^n\right)^2 \int_{-1}^{1}\left[p_n(x)\right]^2 dx = \int_{-1}^{1} D^n\left(x^2-1\right)^n D^n\left(x^2-1\right)^n dx$$

$$= \left|D^n\left(x^2-1\right)^n\ D^{n-1}\left(x^2-1\right)^n\right|_{-1}^{1} - \int_{-1}^{1} D^{n+1}\left(x^2-1\right)^n D^{n-1}\left(x^2-1\right)^n dx$$

$$= -\int_{-1}^{1} D^{n+1}\left(x^2-1\right)^n D^{n-1}\left(x^2-1\right)^n dx$$

$$= (-1)^n \int_{-1}^{1} D^{2n}\left(x^2-1\right)^n\left(x^2-1\right)^n dx \quad \text{[Integrating by parts (n–1) times]}$$

$$= (-1)^n \int_{-1}^{1} (2n)!\ \left(x^2-1\right)^n dx = 2(2n)! \int_{0}^{1}\left(1-x^2\right)^n dx$$

$$= 2(2n)! \int_{0}^{\frac{\pi}{2}} \cos^{2n+1}\theta\ d\theta \quad \text{put } x = \sin\theta = 2(2n)!\ \frac{2n(2n-2)\ ....\ 4\cdot 2}{(2n+1)(2n-2)\ ....\ 2\cdot 1}$$

$$= \frac{2(2n)!\ \left[2n(2n-2)\ .....4\cdot 2\right]^2}{(2n+1)!} = \frac{2}{2n+1}\cdot\left[2^n\cdot n!\right]^2$$

$$\therefore\ \left(n!\ 2^n\right)^2 \int_{-1}^{1}\left[p_n(x)\right]^2 dx = \frac{2}{2n+1}\cdot\left[2^n\ n!\right]^2$$

$$\Rightarrow \int_{-1}^{1}\left[p_x(x)\right]^2 dx = \frac{2}{2n+1}$$

Ans. (b). We know $J_n(x)=\left(\frac{x}{2}\right)^n\left\{\frac{1}{\Gamma(n+1)}-\frac{1}{1!\Gamma(n+2)}\left(\frac{x}{2}\right)^2\right.$

$$\left.+\frac{1}{2!\Gamma(n+3)}\left(\frac{x}{2}\right)^4-\frac{1}{3!\Gamma(x+4)}\left(\frac{x}{2}\right)^6+\ldots\right\}$$

$$\therefore J_{1/2}(x)=\left(\frac{x}{2}\right)^{\frac{1}{2}}\left\{\frac{1}{\Gamma\left(1+\frac{1}{2}\right)}-\frac{1}{1!\Gamma\left(2+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2+\frac{1}{2!\Gamma\left(3+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4-\frac{1}{3!\Gamma\left(4+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^6+\ldots\right\}$$

$$=\left(\frac{x}{2}\right)^{\frac{1}{2}}\left\{\frac{1}{\Gamma\left(\frac{3}{2}\right)}-\frac{1}{1!\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2+\frac{1}{2!\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4-\frac{1}{3!\Gamma\left(\frac{9}{2}\right)}\left(\frac{x}{2}\right)^6+\ldots\right\}$$

$$=\left(\frac{x}{2}\right)^{\frac{1}{2}}\left\{\frac{\cdot}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}-\frac{1}{1!\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2+\frac{1}{2!\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4-\frac{1}{3!\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^6+\ldots\right\}$$

$$=\sqrt{\frac{x}{2}}\cdot\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left\{\frac{2}{1!}-\frac{2x^2}{3!}+\frac{2x^4}{5!}\ \ldots\right\}$$

$$=\sqrt{\frac{2}{x\pi}}\left\{\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}\ \ldots\right\}$$

$$=\sqrt{\frac{2}{x\pi}}\sin x$$

Ans. (c) Cayley Hamilton Theorem : Every square matrix satisfies its own characteristic equation.

The C.E. of the given matrix is

$$\begin{vmatrix}-\lambda & 0 & 1\\ 3 & 1-\lambda & 0\\ -2 & 1 & 4-\lambda\end{vmatrix}=0\Rightarrow\lambda^3-5\lambda^2+6\lambda-5=0$$

$$A=\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix},\quad A^2=\begin{pmatrix}-2 & 1 & 4\\ 3 & 1 & 3\\ -5 & 5 & 14\end{pmatrix},\quad A^3=\begin{pmatrix}-5 & 5 & 14\\ -3 & 4 & 15\\ -13 & 19 & 51\end{pmatrix}$$

$\therefore A^3-5A^2+6A-5I=0$

$\therefore$ A satisfies the theorem.