ENGINEERING MATHEMATICS—2006

Time : 3 Hours *Full Marks : 70*

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
***Note :** Answer the following questions as per direction.*

1. Answer any ten questions from the following : $10 \times 1 = 10$

(i) Rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$ is—(a) 4 (b) 3 (c) 2 (d) 1

Ans. Rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$ Two non zero rows.

$\therefore$ Rank = 2

(ii) The value of t for which the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 5 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$ is singular is—(a) $\frac{3}{2}$ (b) 2 (c) 1 (d) $\frac{1}{3}$

Ans. A matrix T is singular when det (T) = 0

Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 5 & t & 3 \\ 0 & 3 & 1 \end{pmatrix}$ $\therefore$ det (A) = 0

$\Rightarrow 2(t - 9) - 5(-3) = 0$

$\Rightarrow 2t - 18 + 15 = 0$

$\Rightarrow t = \frac{3}{2}$

(iii) The equation $x + y + z = 0$ has—(a) infinite number of solutions (b) no solution (c) unique solution (d) two solutions.

Ans. The equation $x + y + z = 0$ has infinite number of solutions.

(iv) The value of k for which the vectors (1, 2, 1), (k, 1, 1) and (1, 1, 2) are linearly dependent is

(a) 2 (b) $\frac{2}{3}$ (c) -1 (d) $\frac{3}{2}$

Ans. The vectors (1, 2, 1), (k, 1, 1) and (1, 1, 2) are linearly dependent then $\begin{vmatrix} 1 & 2 & 1 \\ k & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0$

$\Rightarrow (2-1) - k(4-1) + 1(2-1) = 0 \quad \Rightarrow k = \frac{2}{3}$

(v) The eigenvalues of the matrix A are a and b ; then the eigenvalues of A^2 are
(a) ab, b^2 (b) a^2, b (c) a^2, b^2 (d) a, b.

Ans. The eigen values of the matrix A are a and b, and let x be the eigen vector.
$\therefore AX = ax$
$\Rightarrow A^2x = a(AX) = a(ax) = a^2x \qquad \therefore a^2$ is the eigenvalue of A^2
Similarly b^2 is the eigenvalue of A^2 $\therefore$ Eigen values of A^2 are a^2 and b^2.

(vi) If a linear transformation $T : R^2 \rightarrow R^2$ be defined by $T(x_1 . x_2) = (x_1 . x_2)$, then Ker (T) is

(a) $\{(-1, -1), (1, 1)\}$ (b) $\left\{(1, 2), \left(1, \frac{1}{2}\right)\right\}$ (c) $\{(1, 0), (0, 1)\}$ (d) $\{(0, 0)\}$

Ans. Let $(x, y) \in \ker(T)$, then $T(x, y) = (0, 0)$
$\Rightarrow (x, y) = (0, 0) \quad \therefore x = 0, y = 0$

(vii) The interpolation formula which can be used to find a polynomial from following data :

x :	0	1	2	4
y :	3	9	17	22

is—(a) Newton's forward interpolation formula (b) Gaussian interpolation formula (c) Lagrange's interpolation formula (d) Newton's backward interpolation formula.

Ans. Lagrange's interpolation formula.

(viii) Which of the following is not true (the notations have their usual meanings)?

(a) $\Delta = E - 1$ (b) $\Delta . \nabla = \Delta - \nabla$ (c) $\frac{\Delta}{\nabla} = \Delta + \nabla$ (d) $\nabla = 1 - E^{-1}$

Ans. (c) $\frac{\Delta}{\nabla} = \Delta + \nabla$, is not true.

(ix) If E_a is the absolute error in a quantity whose true and approximate values are given by x_t and x_a, then the relative error is given by

(a) $\left|\frac{E_a}{x_a}\right|$ (b) $\left|\frac{E_a}{x_t}\right|$ (c) $\left|\frac{E_a}{x_t - x_a}\right|$ (d) $|E_a|$

Ans. Absolute error E_a = true value – approximate value.
$\Rightarrow E_a = X_t - X_a$

Relative Error $= \left|\frac{x_t - x_a}{x_t}\right| = \left|\frac{E_a}{X_t}\right|$

(x) $L\{te^{2t}\}$ is equal to

(a) $\frac{1}{s-2}$ (b) $2(s-2)^2$ (c) $\frac{1}{(s-2)^2}$ (d) $\frac{L^2}{s^2}$

Ans. We know $L\left(t^n\right) = \frac{n!}{n+1}$ $\therefore L(t) = \frac{1}{s^2}$

$L\left\{e^{at}f(t)\right\} = \bar{f}(s-a)$ $\therefore L\left\{t\,e^{2t}\right\} = \bar{f}(s-2)$

When $\bar{f}(s) = L(t) = \frac{1}{s^2}$ $\therefore L\left\{t\,e^{2t}\right\} = \bar{f}(s-2) = \frac{1}{(s-2)^2}$

(xi) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$ is equal to

(a) $\frac{1}{2a}t\sin at$ (b) $\frac{1}{a}t\sin at$ (c) $\frac{1}{a^2}\sin at$ (d) $\cos at$

Ans. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

(xii) $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$. Then $L\left(\frac{\sin at}{t}\right)$ is

(a) $\tan^{-1}\left(\frac{1}{s^2}\right)$ (b) $\tan^{-1}\left(\frac{a}{s}\right)$ (c) $\tan^{-1}\left(\frac{1}{as}\right)$ (d) $\tan^{-1}\left(\frac{1}{s^2+a^2}\right)$

Ans. We know, if $L\{f(t)\} = f(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Given, $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right) \Rightarrow L\{f(t)\} = \bar{f}(s)$, where $f(t) = \frac{\sin t}{t}$, $\bar{f}(s) = \tan^{-1}\frac{1}{s}$

Then $L\left(\frac{\sin at}{t}\right) = aL\left(\frac{\sin at}{at}\right) = aL\left\{\frac{\sin at}{at}\right\} = aL\{f(at)\} = \frac{a}{a}\cdot\bar{f}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)$

(xiii) The general solution of the ordinary differential equation $\frac{d^2y}{dx^2}+4y=0$ (where A and B are arbitrary constants)

(a) $Ae^{2x}+Be^{-2x}$ (b) $(A+Bx)\,e^{2x}$

(c) $A\cos 2x + B\sin 2x$ (d) $(A+Bx)\cos 2x$

Ans. $\frac{d^2y}{dx^2}+4y=0$

$\Rightarrow D^2y + 4y = 0$

$\Rightarrow (D^2 + 4)y = 0$

A.E. $D^2 + 4 = 0$

$D = \pm 2i = 0 \pm 2i$

$\therefore$ The solution $y = (A \cos 2x + Bs \text{ in } 2x)\, e^{0.x} = A \cos 2x + B \sin 2x$

(xiv) The degree and order of the differential equation

$$\left(\frac{d^2y}{dx^2} + 2\right)^{\frac{3}{2}} = x\frac{dy}{dx} \text{ are}$$

(a) degree $= \frac{3}{2}$, order = 2
(b) degree = 2, order = 3
(c) degree = 3, order = 2
(d) degree = 2, order = 1

Ans. $\left(\frac{d^2y}{dx^2} + 2\right)^{\frac{3}{2}} = x\frac{dy}{dx} \Rightarrow \left(\frac{d^2y}{dx^2} + 2\right)^{3} = x^2\left(\frac{dy}{dx}\right)^2$

$\therefore$ The degree is 3 and the order is 2.

2. Answer *any five* questions from the following : 5 × 3 = 15

(i) Prove that

$$\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix} = \left(a^3 + b^3 + c^3 - 3abc\right)^2$$

Ans. We know from Jacobi theorem :

If $A = (a_{ij})_{n \times n}$ be a square matrix and A_{ij} be the cofactor of a_{ij} in det(A), then det (A_{ij}) is said to be the adjoint of det(A).

Jacobi theorem states that det (A_{ij}) = [det (A)]$^{n-1}$

We let $|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -\left(a^3 + b^3 + c^3 - 3abc\right)$

$$\therefore |\text{adjoint of } A| = \begin{vmatrix} bc - a^2 & ac - b^2 & ab - c^2 \\ ac - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix}$$

$= [\det(A)]^2 = (a^3 + b^3 + c^3 - 3abc)^2$

(ii) Prove that $\Delta u = 3x(x - 1)$ when $u = x(x - 1)(x - 2)$, interval of difference being unity and Δ being the forward difference operator.

Ans. $u = x(x - 1)(x - 2)$

$\therefore \Delta u = u(x+1) - u(x)$

$= (x+1)(x)(x-1) - x(x-1)(x-2) = x(x-1)(x+1-x+2) = 3x(x-1)$

(iii) Does $S = \{(x, y, z) ; x + y - z = 0\}$ form a subspace of R^3? Justify.

Ans. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in S$ and $\beta = (\beta_1, \beta_2, \beta_3) \in S$

Then $\alpha_1 + \alpha_2 - \alpha_3 = 0$ and $\beta_1 + \beta_2 - \beta_3 = 0$

$a\alpha + \beta b = (a\alpha_1, a\alpha_2, a\alpha_3) + (b\beta_1, b\beta_2, b\beta_3) = (a\alpha_1 + b\beta_1, a\alpha_2 + b\beta_2, a\alpha_3 + b\beta_3)$

Now, $(a\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2) - (a\alpha_3 + b\beta_3)$

$= a(\alpha_1 + \alpha_2 - \alpha_3) + b(\beta_1 + \beta_2 - \beta_3) = a.0 + b.0 = 0$

$\therefore a\alpha + \beta b \in S$

$\therefore$ S is a subspace of R^3.

(iv) Solve the following differential equation :

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

Ans. $y = \frac{x\,dy}{dx} + \left(\frac{dy}{dx}\right)^2$.

$\Rightarrow p^2 + px - y = 0 \;\ldots\ldots (1) \quad \left[\because \frac{dy}{dx} = p\right]$

Diff. w.r. to x.

$2p\frac{dp}{dx} + p + x\frac{dp}{dx} - \frac{dy}{dx} = 0$

or, $(2p+x)\frac{dp}{d} + p - p = 0$ or, $(2p+x)\frac{dp}{dx} = 0$ $\quad \therefore \frac{dp}{dx} = 0 \Rightarrow p = c$

$\therefore$ From (1) $c^2 + xc - y = 0$ or, $y = cx + c^2$ and $2p + x = 0$

$\Rightarrow p = -\frac{x}{2}$

$\therefore$ From (1) $\frac{x^2}{4} - \frac{x^2}{2} - y = 0 \quad \Rightarrow 4y + x^2 = 0$

$\therefore$ The general solution is $y = cx + c^2$ and singular solution is $4y + x^2 = 0$.

(v) Find $L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol. : $L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left[\frac{1}{b^2-a^2}\left\{\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right\}\right]$

$$=\frac{1}{b^2-a^2}L^{-1}\left[\frac{1}{s^2+a^2}-\frac{1}{s^2+b^2}\right]=\frac{1}{b^2-a^2}\left\{\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]-\frac{1}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right]\right\}$$

$$=\frac{1}{b^2-a^2}\left[\frac{\sin at}{a}-\frac{\sin bt}{b}\right]$$

(vi) If $A(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, **show that** $A(\theta)\,A(\phi)=A(\phi)\,A(\theta)=A(\theta+\phi)$

Sol. : $A(\theta)\,A(\phi)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$

$$=\begin{pmatrix}\cos\theta\cos\phi-\sin\phi\sin\phi & -\cos\theta\sin\phi-\sin\theta\cos\phi\\ \sin\theta\cos\phi-\cos\phi\sin\phi & -\sin\theta\sin\phi+\cos\theta\cos\phi\end{pmatrix}=\begin{pmatrix}\cos(\theta+\phi) & -\sin(\theta+\phi)\\ \sin(\theta+\phi) & \cos(\theta+\phi)\end{pmatrix}=A(\theta+\phi)$$

Similarly $A(\phi)\,A(\theta)=A(\phi+\theta)$

(vii) Find $L\{e^t \sin t\cos t\}$.

Sol. : $L\left\{e^t \sin t\cos t\right\}=\frac{1}{2}L\left\{e^t\sin 2t\right\}=\frac{1}{2}L\left\{e^t f(t)\right\}$, where $f(t)=\sin 2t=\frac{1}{2}\bar{f}(s-1)$

$f(t)=\sin 2t$

$\therefore L\{f(t)=L\{\sin 2t\}$ or, $\bar{f}(s)=\frac{2}{s^2+4}$

$\therefore L\left\{e^t\sin t\cos t\right\}=\frac{1}{2}\cdot\frac{2}{(s-1)^2+4}=\frac{1}{s^2-2s+5}$

(viii) Evaluate $\left(\frac{\Delta^2}{E}\right)x^2$ if the interval of diference is 2, where Δ is the difference operator and E is the shift operator. 8

Sol. : $\left(\frac{\Delta^2}{E}\right)x^2=\Delta^2\left(\frac{1}{E}\left(x^2\right)\right)=\Delta^2\left(E^{-1}\left(x^2\right)\right)=\Delta^2(x-2)^2=\Delta\left\{\Delta\left[(x-2)^2\right]\right\}$

$$=\Delta\left[x^2-(x-2)^2\right]=\Delta(4x-4)=\Delta(4x)-\Delta(4)=\Delta(4x)$$
$$=4\Delta(x)=4(x+2-x)=8$$

3. Answer any five questions from the following : $5\times 3=15$

(i)(a) Find the rank of the rectangular matrix :

$$\begin{pmatrix}1 & 3 & 2 & 4 & 1\\ 0 & 0 & 2 & 2 & 0\\ 2 & 6 & 2 & 6 & 2\\ 3 & 9 & 1 & 10 & 6\end{pmatrix}$$

Sol. : $\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$

$$\xrightarrow[R_4-3R_2]{R_1-2R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & -3 \end{pmatrix} \xrightarrow[R_4+R_2]{R_3+R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & -3 \end{pmatrix}$$

There are 3 non zero rows. $\therefore$ Rank = 3

(b) **Show that every square matrix can be expressed as a sum of a symmetric and skew-symmetric matrix.** 5 + 4 = 9

Sol. : Let A be the square Matrix.

$$\therefore A = \frac{1}{2}\cdot 2A = \frac{1}{2}\left[A + A^T + A - A^T\right]$$

$$= \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) = P + Q$$

Where $P = \frac{1}{2}\left(A + A^T\right)$ and $Q = \frac{1}{2}\left(A - A^T\right)$

$$\therefore P^T = \frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left(A^T + A\right) = P$$

$\therefore$ P is symmetric matrix.

and $Q^T = \frac{1}{2}\left(A - A^T\right)^T = \frac{1}{2}\left(A^T - A\right) = -\frac{1}{2}\left(A - A^T\right) = -Q$ $\therefore Q = -Q^T$

$\therefore$ Q is a skew symmetric matrix.

$$\therefore A = P + Q = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

(ii) (a) Solve, if possible :

$x + y + z = 1$
$2x + y + 2z = 2$
$3x + 2y + 3z = 5$

Sol. : The co-efficient matrix A and the augmented metrix B of the equation are

$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix}$ $k = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 3 & 5 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 3 & 5 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 \end{pmatrix} \xrightarrow{R_3+R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

Rank of A = 2, Rank of B = 3 $\therefore$ Rank of A $\neq$ Rank of B

$\therefore$ The equations have no solution.

(b) If a matrix A is invertible and its eigenvalues are $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ and $B = A^{-1}$, show that the eigenvalues of B are

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots\ldots, \frac{1}{\lambda_n}$$ 4 + 5 = 9

Sol. : Since λ_i, i = 1, 2, ... n are the eigen values of the matrix A of order n.

$\therefore |A - \lambda_i I_n| = 0$

Now, $|A^{-1} - \lambda_i^{-1} I_n| = \frac{1}{\lambda_i^n} |(\lambda_i A^{-1} - I_n)| = \frac{1}{\lambda_i^n \det(A)} |(\lambda_i A^{-1} - I_n)A| = \frac{1}{\lambda_i^n |A|} |\lambda_i I_n - A| = 0$

This proves that λ_i^{-1} are the eigen values of A^{-1}.

(iii) (a) Prove that the set of all second order real square matrices is a vector space with respect to matrix addition and multiplication by a real number.

(b) A linear transformation $T : R^3 \rightarrow R^2$ is defined by $T(x, y, z) = (x + y, x - z)$. Find its rank and nullity.

4 + 5 = 9

(iv) (a) Find the interpolating polynomial which corresponds to the following data :

x :	–1	–2	2	4
f(x) :	1	–9	11	69

Hence find f(0).

(b) Find f(x) where $\Delta f(x) = x^2 + 11x + 5$, interval of difference being unity and Δ is the forward difference operator. 5 + 4 = 9

(v) (a) Evaluate $\int_{0.1}^{0.7} (e^x + 2x)\, dx$ by Simpson's one-third rule, taking 6 intervals correct to 5-decimal places. Write the general error term in simple Simpson's one-third rule, in terms of the numbers of intervals (n) and the spacings (h).

4 + 1 = 5

(b) Find $L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\}$

(vi) (a) Find L{F(t)}, where

$F(t) = 1$ if $t > a$

$= 0$ if $t < a$

where a is any positive real number.

(b) Solve the following differential equation, using Laplace and inverse Laplace Transforms :

$(D^2 - 1)y = a \cos h\, nt$

where $y(0) = 0$, $y^1(0) = 2$

(vii) (a) Solve the following differential equation :

$$\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$$

(b) Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2}+4y=4\sec^2 2x.$$ 4 + 5 = 9

(viii) (a) Solve : $e^x \sin y\, dx + (e^x + 1)\cos y\, dy = 0$.

(b) State the convolution theorem of inverse Laplace Transform and apply the theorem to evaluate

$$L^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}$$ 4 + 5 = 9

Sol. : 3.(iii)(a). Let V be the set of all second order real matrices and A, B, C$\in$V.

Now, (i) A + B is a second order matrix. $\therefore$ A + B$\in$V

(ii) A + B = B + A $\because$ Matrix addition is commulative.

(iii) A + (B + C) = (A + B) + C $\because$ Matrix addition is associative.

(iv) $A + O_{2\times 2} = A$ and $O_{2\times 2} \in V$

(v) A – A = θ

$\Rightarrow$ A + (–A) = 0

$\Rightarrow$ A + D = 0

when $D = -A = \left(-a_{ij}\right)_{2\times 2} \in V$

$\therefore$ For each A there exist D $\in$V such that $A + D = O_{2\times 2}$

(vi) $CA = (Ca_{ij})_{2\times 2}$ is a second order matrix.

$\therefore$ CA$\in$V

(vii) $c(dA) = c(da_{ij})_{2\times 2} = (cd\, a_{ij})_{2\times 2} = cdA$

(viii) $(c + d)A = [(c + d)a_{ij}]_{2\times 2} = (ca_{ij})_{2\times 2} + (da_{ij})_{2\times 2} = cA + dA$

(ix) $c(A + B) = c(a_{ij} + b_{ij}) = (ca_{ij})_{2\times 2} + (cb_{ij})_{2\times 2} = cA + cB$

(x) $1 \cdot A = (1a_{ij})_{2\times 2} = (a_{ij})_{2\times 2} = A$

$1 \in R$

$\therefore$ The set V forms a vector space.

Soln. : 3.(iii)(b).

Rank of T = dim (I^m T)

Nullify of T = dim (ker T)

Let $(x, y, z) \in \ker(T)$

Then $T(x, y, z) = (0, 0)$

or, $(x + y, x - z) = 0, 0$ or, $x + y = 0$ and $x - z = 0$

The solution is $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-1} = k$ $\therefore (x, y, z) = k(-1, 1, -1), k \in R$

$\therefore \ker(T) = L\{(-1, 1, -1)\}$ $\therefore \dim(\ker T) = 1$ $\therefore$ Nullity = 1

we know $\dim(\ker T) + \dim(I^m T) = \dim R^3$

$= 1 + \dim(I^m T) = 3 = \dim(I^m T) = 3 - 1 = 2$ $\therefore$ Rank = 2

Soln. : 3.(iv)(a) We know from Lagrange's in terpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x+2)(x-2)(x-4)}{(-1+2)(-1-2)(-1-4)} \times 1 + \frac{(x+1)(x-2)(4-4)}{(-2+1)(-2-2)(-2-4)}$$

$$\times(-9) + \frac{(x+1)(x+2)(4-4)}{(2+1)(2+2)(2-4)} \times 11 + \frac{(x+1)(x+2)(x-2)}{(4+1)(4+2)(4-2)} \times 69$$

$$= \frac{(x^2-4)(x-4)}{15} + \frac{(x^2-3x-4)(x-2) \cdot 3}{8} - \frac{(x^2-3x-4)(x+2) \times 11}{24} + \frac{(x^2-4)(x+1)}{60} \times 69$$

$$f(x) = \frac{x^2-4}{15} \frac{(4x-16+69x+64)}{4} + \frac{x^2-3x-4}{8} \frac{(9x-18-11x-22)}{3}$$

$$= \frac{68x^3-32x^2+28x+188}{60} \qquad \therefore f(0) = \frac{188}{60} = \frac{47}{15}$$

Soln. : 3.(iv)(b). $\Delta f(x) = x^2 + 11x + 5$ .... (1)

Since $\Delta f(x)$ is a polynomial of degree 2

Let f(x) is a polynomial of degree 3.

Let the form of $f(x) = ax^3 + bx^2 + cx + d$ $\therefore \Delta f(x) = f(x+1) - f(x)$

$= a(x+1)^3 + b(x+1)^2 + c(x+1) + d - (ax^3 + bx^2 + cx + d)$

$= ax^3 + (3a+d)x^2 + (3a+2b+c)x + c + a + b + d - (ax^3+bx^2+cx+d)$

$= 3ax^2 + (3a+2b)x + a + b + c$ ...... (ii)

Comparing (i) and (ii)

3a = 1 .... (iii)

3a + 2b = 11 .... (iv)

a + b + c = 5 .... (v) $\qquad \therefore a = \frac{1}{3}$

$b = \frac{1}{2}(11 - 3a) = \frac{1}{2}(11 - 1) = 5$

$c = 5 - a - b = 5 - \frac{1}{3} - 5 = -\frac{1}{3} \qquad \therefore f(x) = \frac{1}{3}x^3 + 5x^2 - \frac{1}{3}x + d$

Another Method : We first express $\Delta f(x)$ in term of factorial notation.

1	1	11	5
	0	1	
	1	12	
2	0		
	1		

$\therefore \Delta f(x) = x^{(2)} + 12\,x^{(1)}$

$\therefore f(x) = \frac{1}{3}x^{(3)} + \frac{12}{2}x^{(2)} + \frac{5}{1}\cdot x^{(1)} + d = \frac{1}{3}x(x-1)(x-2) + 6x(x-1) + 5.x + d$

$= \frac{1}{3}x^3 + 5x^2 - \frac{1}{3}x + d$

Soln. : 3.(v)(a).

Let $I = \int_{0.1}^{0.7}\left(e^x + 2x\right)dx$

Here $x_0 = 0.1$, $n = 6$, $x_6 = 0.7$, $h = \frac{x_6 - x_0}{n} = \frac{0.7 - 0.1}{6} = 0.1$

$f(x) = e^x + 2x$

$\therefore$

x	f(x)
0.1	1.3051709
0.2	1.6214028
0.3	1.9498588
0.4	2.2918247
0.5	2.6487213
0.6	3.0221188
0.7	3.4137527

We know from Simpson's $\frac{1}{3}$ rule

$$I=\frac{h}{3}\left[y_0+4\left(y_1+y_3+y_5\right)+2\left(y_2+y_4\right)+y_6\right]$$

$=\frac{0\cdot 1}{3}$ [1.3051709 + 4 (1.6214028 + 2.2918247 + 3.0221188) + 2(1.9498588 + 2.6487213) + 3.4137527] = 1.3885823

Error term in Simpson's $\frac{1}{3}$ rule is $-\frac{nh^5}{180}y^{iv}(\zeta)$, where $x_0<\zeta<x_n$

Soln. : 3.(v)(b). $L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\}=L^{-1}\left\{\frac{1}{(s+1)^3}-\frac{2}{(s+1)}+\frac{1}{(s+1)^5}\right\}$

$$=\frac{1}{2!}e^{t}\cdot t^2-\frac{2}{3!}e^{t}\cdot t^3+\frac{1}{4!}e^{t}\cdot t^4=e^{-t}\left[\frac{t^2}{2}-\frac{t^3}{3}+\frac{t^4}{24}\right]$$

Another Method : If $\bar{f}(s)=\frac{4!}{(s+1)^5}$ then $f(t) = e^{-t}\,t^4$

If $\bar{f}(s)=\frac{1}{(s+1)^5}$ then $f(t)=\frac{e^{-t}t^4}{4!}=\frac{e^{-t}t^4}{24}$

$\therefore$ We know $L^{-1}\left\{s^n\bar{f}(s)\right\}=\frac{d^n}{dt^n}f(t)$

$$\therefore L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\}=L^{-1}\left\{s^2\cdot\frac{1}{(s+1)^5}\right\}=\frac{d^2}{dt^2}\left[\frac{e^{-t}t^4}{24}\right]$$

$$=\frac{1}{24}\left[12\,t^2e^{-t}-8t^3e^{-t}+t^4e^{-t}\right]=e^{-t}\left[\frac{t^2}{2}-\frac{t^3}{3}+\frac{t^4}{24}\right]$$

Soln. : 3.(vi)(a). $L\{F(t)\}=\int_0^{\alpha}e^{-st}F(t)dt=\int_0^{a}e^{-st}F(t)\,dt+\int_a^{\alpha}e^{-st}F(t)\,dt$

$$=0+\int_a^{\alpha}e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_a^{\alpha}=\left[\frac{1}{-se^{st}}\right]_a^{\alpha}=-0+\frac{1}{se^{as}}=\frac{e^{-as}}{s}$$

Soln. : 3.(vi)(b). $(D^2-1)y = a\cosh nt$

Taking the laplace transforms on the both sides

$$s^2\bar{y}-sy(0)-y'(0)-\bar{y}=a\cdot\frac{s}{s^2-n^2}$$

or, $\left(s^2-1\right)\bar{y}-2=\frac{as}{s^2-n^2}$ or, $\left(s^2-1\right)\bar{y}=2+\frac{as}{s^2-n^2}$ or, $\bar{y}=\frac{2}{s^2-1}+\frac{as}{\left(s^2-n^2\right)\left(s^2-1\right)}$

or, $\bar{y}=2\cdot\frac{1^2}{s^2-1^2}+a\cdot\left[\frac{s}{s^2-n^2}-\frac{s}{s^2-1}\right]\cdot\frac{1}{n^2-1}$ or, $\bar{y}=2\cdot\frac{1^2}{s^2-1^2}+\frac{a}{n^2-1}\left[\frac{s}{s^2-n^2}-\frac{s}{s^2-1}\right]$

Taking Laplace inverse transforms $y=2L^{-1}\left\{\frac{1^2}{s^2-1^2}\right\}+\frac{a}{n^2-1}L^{-1}\left\{\frac{s}{s^2-n^2}-\frac{s}{s^2-1}\right\}$

$$=2\cdot\sinh t+\frac{a}{n^2-1}\left(\cosh nt-\cosh t\right)$$

Soln. : 3.(vii)(a). $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$ or, $p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x}$ $\left[\because p=\frac{dy}{dx}\right]$

or, $p^2-\left(\frac{x}{y}-\frac{y}{x}\right)p-1=0$ or, $\left(p-\frac{x}{y}\right)\left(p+\frac{y}{x}\right)=0$ or, $p-\frac{x}{y}=0$ and or, $p+\frac{x}{y}=0$

or, $\frac{dy}{dx}-\frac{x}{y}=0$ or, $\frac{dy}{dx}+\frac{y}{x}=0$ or, $y\,dy-x\,dx=0$ or, $\frac{dy}{y}+\frac{dx}{x}=0$

Integrating $y^2-x^2-c_1=0$ Integrating $xy-c_2=0$

$\therefore$ The general solution : $(y^2-x^2-c_1)(xy-c_2)=0$

Soln. : 3.(vii)(b) $\frac{d^2y}{dx^2}+4y=4\sec^2 2x$

To find C.F.

Auxiliary equation is $m^2+4=0 \Rightarrow m=\pm 2i$

$\therefore$ C. F. is $c_1\cos 2x+c_2\sin 2x=c_1y_1+c_2y_2$ $[y_1=\cos 2x\,;\,y_2=\sin 2x]$

To find P. F. $W=\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}=2$

$$\therefore\ \text{P.I.}=y_1\int\frac{-\sin 2x\,.\,4\sec^2 2x}{2}dx+y_2\int\frac{\cos 2x.4\sec^2 2x}{2}dx$$

$$=-2\cos 2x\int\sec 2x\tan 2x\,dx+2\sin 2x\int\sec 2x\,dx$$

$$=\frac{-2\cos 2x.\sec 2x}{2}+2\sin 2x\cdot\frac{1}{2}\log(\sec 2x+\tan 2x)$$

$$=-1+\sin 2x\log(\sec 2x+\tan 2x)$$

$\therefore$ The solution is $y=c_1\cos 2x+c_2\sin 2x-1+\sin 2x\,.\,\log(\sec 2x+\tan 2x)$

Soln. : 3.(vii) (a) $e^x \sin y\, dx + (e^x + 1) \cos y\, dy = 0$

or, $\frac{e^x}{e^x+1} dx + \frac{\cos y}{\sin y} dy = 0$ or, $\frac{d(e^x+1)}{e^x+1} + \frac{d(\sin y)}{\sin y} = 0$

Integrating, $\log(e^x + 1) + \log(\sin y) = \log c$ or, $\sin y \,.\, (e^x + 1) = c$

Soln. : 3.(viii)(b). Convolution theorem of inverse laplace transform.

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{g(s)\} = g(t)$

then, $L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du$

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right\}$$

Let $\bar{f}(s) = \frac{s}{s^2+1}$ then $f(t) = \cos t$ and $\bar{g}(s) = \frac{1}{s^2+1}$ then $g(t) = \sin t$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \int_0^t \cos u \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\sin t + \sin(t-2u)]\, du$$

$$= \frac{1}{2}\left[\sin t \,.\, u + \frac{\cos(t-2u)}{2}\right]_0^t$$

$$= \frac{1}{2}\left[t \sin t + \frac{\cos t}{2} - 0 - \frac{\cos t}{2}\right] = \frac{t \sin t}{2}$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{t \sin t}{2}$$