MATHEMATICAL SCIENCES
PAPER-II

1. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. Prove or disprove each of the statements :

   1. If $\{x_n y_n\}$ converges, and if $\{y_n\}$ is convergent, then $\{x_n\}$ is convergent.
   2. $\{x_n + y_n\}$ converges to x+y if $\{x_n\}$ converges to x and $\{y_n\}$ converges to y.
   3. If $\{x_n/y_n\}$ is convergent, then both $\{x_n\}$ and $\{y_n\}$ are convergent.
   4. If $\{x_n\}$ is convergent and $\{y_n\}$ is divergent, then $\{x_n y_n\}$ is divergent.

2. Let $f : [a,b] \to \mathbb{R}$ be differentiable.
   (a) Prove that
   $$\int_a^b f^2(t)\,dt = 0 \text{ iff } f \equiv 0 \text{ on } [a,b]..$$
   (b) If $\int_a^b f^3(t)\,dt = 0$, then for some $t_0 \varepsilon [a,b]$, either $f'(t_0) = 0$ or $f(t_0) = 0$.

3. Let $\{f_n\}$ be a sequence of real-valued Lebesgue integrable functions on $\mathbb{R}$ such that $\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f_n| < \infty$. Prove that for any $\alpha \in \mathbb{R}$, $\sum_{n=1}^{\infty} e^{in\alpha} f_n(x)$ converges a.e. to a function $g_\alpha(x)$ and that g_α is Lebesgue integrable.

4. Let $a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ and let $f(x) = e^{x.a}$ where $x = (x_1, x_2 \cdots, x_n) \in \mathbb{R}^n$, and $x.a = \sum_{i=1}^{n} x_i.a_i$. Compute the directional derivative of f at a point $p \in \mathbb{R}^n$ in the direction of $h \in \mathbb{R}^n$.

5. (a) Let $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \leq 37\} \cap \{(x,y) \in \mathbb{R}^2 \mid e^x \geq y\}$

   Prove that A is compact.

   (b) Show that any convex subset C of $\mathbb{R}^n$ is connected.

6. Show that the set $\{\log p \mid p \text{ prime number}\}$ is linearly independent over $\mathbb{Q}$.

7. Let V be the vector space of all polynomial functions of degree $< n$, $n \geq 2$, and let $D : V \to V$ denote the derivative map $P \mapsto P'$ on V. Show that D is nilpotent and that D is not diagonalizable.

8. Let A and B be two 3×3 complex matrices. Show that A and B are similar if and only if $\chi_A = \chi_B$ and $\mu_A = \mu_B$, where χ_A, χ_B are characteristic polynomials of A, B, respectively and μ_A, μ_B are minimal polynomials of A, B, respectively.

9. Determine whether the quadratic form $q(X_1,X_2) = 7X_1 + 88X_1X_2 + 88\pi\ X_2^2$ is degenerate or not.

10. Let $\varphi : \Omega \to £$ be a continuous function and suppose $\{z : |z| \leq 1\} \subset \Omega$. Prove that the function defined as $f(z) = \int\limits_{|w|=1} \frac{\varphi(w)}{w-z} dw$ is an analytic function on the open unit disc.

11. Let $f : £ \to £$ be an entire function and that $|f'(z)| < |f(z)|$ for all $z \in £$. Identify all such functions.

12. (a) Let $f : £ \to £$ be an analytic function and $g : £ \to ¡$ a harmonic function. Prove that $g \circ f$ is a harmonic function.

(b) Prove that the function $u(x,y) = e^{2xy}.cos(x^2 - y^2),\ (x,y) \in ¡^2$ is harmonic, and find the harmonic conjugate.

13. Compute $\int\limits_{|z|=2} e^{e^{\frac{1}{z}}} dz$, where the circle is parametrized by $t \to 2e^{it}$, $0 \leq t \leq 2\pi$.

14. Let φ and μ denote the Euler totient function and Möbius function respectively. Show that $n = \sum_{d/n} \varphi(d)$. Hence show that $\frac{\varphi(n)}{n} = \sum_{k/n} \frac{\mu(k)}{k}$.

15. Define conjugacy class in a finite group G and show that the cardinality of any conjugacy class divides the order of G. Use this to show that if p is a prime and G is a group of order p^n, then the centre of G contains elements other than the identity.

16. Let I, I', J be ideals in a commutative ring A. If I, J are comaximal, i.e., $I + J = A$ and I', J are comaximal, i.e., $I' + J = A$, then show that $I I'$ and J are also comaximal.

17. Let $L|K$ be a finite field extension of prime degree p. Show that $L = K[\alpha]$ for any $\alpha \in L \backslash K$.

18. Solve the BVP by determining the appropriate Green's function, expressing the solution as a definite integral.

$-y'' = f(x),\quad y(0) + y'(0) = 0,\quad y(1) + y'(1) = 0.$

19. Consider the initial value problem :
$y' = f(x,y), \quad y(0) = 1$
where $f(x,y) := |xy| + y^2$, $(x,y) \in D$ with $D := [-2,2] \times [-1,3]$.

Show that f (x,y) is bounded and statisfies a Lipschitz condition with respect to y on D and determine a bound and Lipschitz constant on D. Further, determine h, as required in the Picard's Theorem, for a unique solution of the initial value problem to exist on $|x| \le h$.

20. Outline briefly the three classes of integrals of the non-linear first order partial differential equation $f(x,y,z,p,q) = 0$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.
For the partial differential equation
$pqz = p^2(3p^2 + qx) + q^2(py + 4q^2)$, obtain one of the integrals and indicate the procedure for determining the remaining two integrals.

21. Classify and reduce the second order partial differential equation
$u_{xx} - 4x^2 u_{yy} = \frac{1}{x} u_x$ into canonical form and hence, find the general solution.

22. Derive Simpson's $\frac{1}{3}$rd rule to evaluate the integral $\int_a^{a+2h} f(x)dx$. Estimate the error.

23. Find the eigenvalues and the eigenfunctions of the functional
$J(y) = \int_0^1 (y^2 + y'^2)dx$ subject to the conditions $y(0) = y(1) = 0$, $\int_0^1 y^2 dx = 1$.

24. Find the resolvent kernel for the integral equation
$\varphi(s) = f(s) + \lambda \int_{-1}^{1} (st + s^2t^2)\varphi(t)dt$

25. Show that the transformation
$Q = \tan^{-1}\left(\frac{2q}{p}\right)$, $P = q^2 + \frac{1}{4}p^2$ is canonical. Find a generating function.

26. Let X and Y be two independent random variables such that X is uniformly distributed on [0, 1] and Y has a discrete uniform distribution on
$\{0, 1, 2, \mathrm{L}\ , n-1\}$, that is,

$$P(Y = k) = \begin{cases} \frac{1}{n}, & \text{if } k = 0, 1, \mathrm{L}\ , n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Define Z = X + Y. Show that Z is uniformly distributed on $[0, n]$.

27. Let $M(\cdot)$ denote the moment generating function of the standard normal distribution. Let $I(a) = \sup\{ta - \log M(t): t \in \mathbb{R}\}$.
   (i) Find $I(\cdot)$
   (ii) Express $\log M(\cdot)$ in terms of $I(\cdot)$

28. Using the central limit theorem for appropriate Poisson random variables show that

$$\lim_{n\to\infty} e^{-n} \sum_{j=0}^{n} \frac{1}{j!} n^j = \frac{1}{2}.$$

29. Let $\{X_n\}$ be a Markov chain with transition probability matrix P given by

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/5 & 4/5 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Let $p_{ij}^n = P(X_n = j | X_0 = i)$. Find $\lim_{n\to\infty} p_{ij}^n$ for all i,j.

30. A coin with probability p for head is tossed. If a tail turns up, a random number of balls are added to an urn. (Assume that the urn is initially empty). This procedure is repeated till a head appears at which stage it is stopped. Let N denote the number of stages when balls are added, and X_i = number of balls added at i^{th} stage. Assume that $\{X_i\}$ are i.i.d. Poisson (λ) random variables, and that N and $\{X_i\}$ are independent. Find the expected number of balls in the urn when the procedure terminates.

31. Let $x_1, x_2 \cdots, x_n$ be the values of a variable x. Define $x_{max} = \max\{x_1, \cdots, x_n\}$, $x_{min} = \min\{x_1, \cdots, x_n\}$, $R = x_{\max} - x_{\min}$ and $s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 / n.$

Show that $\dfrac{R^2}{2n} \le s^2 \le \dfrac{R^2}{4}$.

32. Let T be the minimum variance unbiased estimator (MVUE) of θ. Then prove that T^K (K a +ve integer) is the MVUE for $E(T^K)$ provided $E(T^{2K}) < \infty$.

33. Suppose $(x_1, y_1), \cdots, (x_n, y_n)$ represent a random sample from $N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho)$. Suppose $\rho = \rho_0$ (known), then find a confidence interval of σ_1/σ_2 with confidence coefficient $(1 - \alpha)$ that incorporates the information that $\rho = \rho_0$.

34. Let $X_1, X_2, \cdots, X_n$ be i.i.d. with density

$$f(x, \theta) = \frac{\theta}{x^2},\ x > \theta,\ \theta > 0.$$

   (a) Find MLE of θ
   (b) Derive the likelihood ratio test for $H_0: \theta = 1$ vs $H_1: \theta \neq 1$.
   (c) If n=4 and the observations are $X_1 = 3.2$, $X_2 = 4.0$, $X_3 = 2.0$, $X_4 = 5.6$, find the P-value of the test derived in (b).

35. Let $X_1, \ldots, X_n$ be independent random variables with common probability distribution function

$$P[X_i \leq x;\ \alpha, \beta] = \begin{cases} 0 & if \quad x < 0 \\ (\frac{x}{\beta})^{\alpha} & if \quad 0 \leq x \leq \beta \\ 1 & if \quad x > \beta \end{cases}$$

where $\alpha, \beta > 0$.

(a) Find a two dimensional sufficient statistic for (α, β)

(b) Find an unbiased estimator of $\frac{1}{\alpha+1}$ when $\beta=1$.

36. Consider a regression model $Y_i = \theta_0 + \theta_1 x_i + \varepsilon_i$, $i=1, \ldots n$, where

$$x_i = \begin{cases} 1 & if \quad i = 1, \ldots, n_1, \\ 0, & if \quad i = n_1 + 1, \ldots, n \end{cases}$$

and ε_i are uncorrelated random errors with mean 0 and common variance σ^2. Let T_1 and T_2 be the two estimators of θ_1 given by $T_1 = Y_1 - Y_n$ and

$T_2 = \overline{Y_1} - Y_n$ where $\overline{Y_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} Y_i$.

(a) Verify whether T_1 and T_2 are unbiased and find their variances.

(b) If possible, propose an unbiased estimator of θ_1, which has variance smaller than that of T_1 and T_2, with justification.

37. Consider a linear model $\underline{Y} = X\underline{\theta} + \underline{\varepsilon}$ where $\underline{Y}$ is a 4×1 vector of observations, $\underline{\theta} = (\theta_1, \theta_2, \theta_3)^T$ is a vector of unknown parameters,

$$X_{4\times3} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and $\underline{\varepsilon}$ is a 4×1 vector of uncorrelated random errors with mean 0 and variance σ^2.

(a) Verify whether the following parametric functions are estimable
(i) $\theta_1 + \theta_2$, (ii) $\theta_1 + \theta_2 + \theta_3$

(b) Find the best linear unbiased estimator(s) of the estimable parametric function(s) in (a) above and obtain the variance of the estimator(s).

38. Suppose $\underset{\sim}{x}_{(p\times 1)} = \begin{pmatrix} \underset{\sim}{x}_{1\,(p_1\times 1)} \\ \underset{\sim}{x}_{2\,(p_2\times 1)} \end{pmatrix} \sim N_p\left(\begin{pmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix}, \Sigma\right)$ with $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} > 0$.

Prove that the necessary and sufficient condition for $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ to be independent is $\Sigma_{12} = 0$. You may assume $\underset{\sim}{x}_i \sim N_{p_i}(\mu_i, \Sigma_{ii})$, $i = 1, 2$.

39. Suppose the problem is to classify an observation $\underset{\sim}{x}$ into one of the populations P_i, $i = 1, 2$. Suppose $f_i(\underset{\sim}{x})$ denotes the density of $\underset{\sim}{x}$ corresponding to population P_i. Also we attach the prior probability p_i ($i = 1, 2$) for an observation $\underset{\sim}{x}$ to belong to population P_i. Find the total probability of misclassification (TPM) and prove that the classification rule minimizing TPM is given by:

for an $\underset{\sim}{x}$, if $\dfrac{f_1(\underset{\sim}{x})}{f_2(\underset{\sim}{x})} \geq \dfrac{p_2}{p_1}$ classify it as an observation belonging to population P_1 and otherwise belonging to population P_2.

40. An unknown number N of taxis plying in a town are supposed to be serially numbered from 1 to N. If the n different taxis you have come across in the town can be assumed to form a simple random sample with replacement, find an unbiased estimator of the total number of taxis in the town. Also find the variance of your estimator.

41. Show that in a randomized block design the estimates of the elementary block and treatment contrasts are orthogonal observational contrasts.

42. Suppose in a 2^5–factorial experiment with factors A, B, C, D and E, a replicate is divided into four blocks of size eight each. How many effects will be confounded? Is it possible to confound the effects AB, BC and ABC? Justify your answer.

43. The daily demand for a commodity is approximately 100 units. Every time an order is placed, a fixed cost of Rs.10,000/- is incurred. The daily holding cost per unit inventory is Rs.2/-. If the lead time is 15 days, determine the economic lot size and the reorder point. Further suppose that the demand is actually an approximation of a probabilistic distribution in which the daily demand is normal with mean $\mu = 100$ and s.d. $\sigma = 10$. How would you determine the size of the buffer stock such that the probability of running out of stock during lead time is at most 0.01?

44. Consider the following linear program (LP)

max $\quad z = 4x_1 + 14x_2$

Subject to

$$\begin{aligned} 2x_1 + 7x_2 + x_3 &= 21 \\ 7x_1 + 2x_2 + x_4 &= 21 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Each of the following cases provides an inverse matrix and its corresponding basic variables for the LP above. Determine whether or not each basic solution is feasible. Interpret these basic feasible solutions and hence find an optimal solution. Is the optimal solution unique?

(a) $\quad (x_2, x_4); \begin{pmatrix} \frac{1}{7} & 0 \\ -\frac{2}{7} & 1 \end{pmatrix}$

(b) $\quad (x_1, x_4); \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{7}{2} & 1 \end{pmatrix}$

(c) $\quad (x_2, x_1); \begin{pmatrix} \frac{7}{45} & -\frac{2}{45} \\ -\frac{2}{45} & \frac{7}{45} \end{pmatrix}$

45. Consider an M/M/c queuing system with parameters λ and μ. Draw its state-transition rate diagram and find the steady-state probability distribution for number of customers in the system.