Instructions:

- The paper is given in two parts: of these, Part A contains 25 multiple-choice questions. **Unless your performance in Part A is satisfactory, your answers to questions in Part B will not be evaluated.**
- Answers for Part A must be entered on the computer-readable answer sheet that is provided separately. Enter at most one choice for each question. Each correct answer carries 3 marks, each incorrect answer carries -1 mark. Unattempted questions fetch 0 marks.
- Part B contains six questions.

PART A

1. Programming languages (such as C) do not predefine numbers like π (the ratio of a circle's circumference to its diameter) or e (the base of the natural logarithm) because:

   (A) these numbers are not constant

   (B) fractions like 22/7 have recurring decimal expansions

   (C) they can be defined using available constants like MAXINT

   (D) the precision available for real numbers across machines is not fixed

2. Given a set X, the powerset of $X \times X$:

   (A) is a relation on X

   (B) includes all binary relations on X

   (C) has all subsets of X

   (D) is uncountable

3. Let f be a function with domain X and range $\wp(X)$, the powerset of X. The set $\{x \mid x \notin f(x)\}$:

   (A) is not well defined

   (B) cannot be in the range of f

   (C) can be in the range of f in general

   (D) can be in the range of f since f can be surjective

4. The set $\{x \in \mathcal{Q} | \sqrt{2} < x < \sqrt{3}\}$ (where $\mathcal{Q}$ is the set of rational numbers)

   (A) has an upper bound but no least upper bound

   (B) has a least upper bound

(C) has no upper bounds

(D) is finite

5. The sequence $1, -1, 1, -1, \ldots$

   (A) has a limit, it is either 1 or -1

   (B) is unbounded and hence has no limit

   (C) is bounded and hence the limit is 1

   (D) is bounded but has no limit

6. We want to show that $lim_{n\to\infty} x_n = 0$ where $x_n = \frac{1}{n}$. Let $\epsilon > 0$. We should:

   (A) find an N such that $|x_n - 0| < \epsilon$, for all $n \geq N$

   (B) show that $|x_n - 0| < \epsilon$, for all n

   (C) show that $\frac{1}{\infty} = 0$

   (D) show that $\frac{1}{n} = 0$, for some sufficiently large n

7. Given the relation $R = \{(1,2),(2,3)\}$ on the set $A = \{1,2,3\}$, the minimum number of ordered pairs that needed to be added to R get an equivalence relation is:
   (A) 5 (B) 6 (C) 7 (D) 8

8. A binary relation R is said to be *transitive* if its composition with itself, R^2, is included in R. It is said to be 3-*transitive* if for every x, y, z, w, whenever xRy and yRz and zRw, then xRw. Which of the following is true?

   (A) All 3-transitive relations are transitive.

   (B) There is no 3-transitive relation which is transitive.

   (C) 3-transitivity is a necessary condition for transitivity.

   (D) 3-transitivity is a sufficient condition for transitivity.

9. On pairs of real numbers, define the relation $(a,b)R(c,d)$ if and only if $2a - b = 2c - d$. Then R is:

   (A) an equivalence relation

   (B) symmetric but not transitive

   (C) transitive but not reflexive

   (D) reflexive and symmetric but not transitive

10. Two positive integers k, l are said to be *multiplicatively dependent* if there are no integers p, q such that $k^p = l^q$. This is the same as saying

    (A) k, l are prime

(B) $\log k - \log l$ is irrational

(C) $\log k / \log l$ is not an integer

(D) $\log k / \log l$ is irrational

11. The total number of terms in the expansion of $(x+y)^{100} + (x-y)^{100}$ after simplification is:

(A) 50

(B) 51

(C) 202

(D) none of these

12. If a polygon has 44 diagonals, then the number of its sides is:

(A) 8

(B) 10

(C) 11

(D) none of these

13. In a strange country, no two persons have the same set of teeth. Assuming that everyone has at least one tooth and no one has more than 32 teeth (and disregarding the shape and size of the teeth), the maximum population of the country is:

(A) $2^{32} - 1$

(B) 2^{32}

(C) 32^{32}

(D) $32!$

14. The number of pairs (A, B) of distinct subsets of $\{1, \ldots, n\}$ with $A \subsetneq B$ is

(A) 4^n

(B) $3^n - 2^n$

(C) $4^n - 3^n$

(D) $2^n + 3^n$

15. There are two bags, each of which contains n balls. The number of ways in which we can select an equal number of balls from both the bags with at least one ball from each bag is:

(A) $(2n)!/n!$

(B) $\binom{n}{n}^2$

(C) $\binom{2n}{n}$

(D) $\binom{2n}{n} - 1$

16. From the set $\{1, 2, 3, 4, 5\}$, two numbers are chosen at random, one by one, without replacement. The probability that the minimum of the chosen two numbers is less than 4 is:

    (A) 2/3
    (B) 0.8
    (C) 0.9
    (D) none of these

17. Let x be a 5-digit number which is a random permutation of the numbers 1, 2, 3, 4 and 5. The probability that x is divisible by 6 is:

    (A) 1/6
    (B) 1/3
    (C) 0.4
    (D) 0.5

18. A tree is a connected acyclic graph. A leaf is a vertex of degree 1. Suppose a tree has a vertex of degree $k \geq 3$. Then T has

    (A) exactly 2 leaves
    (B) at most $k/2$ leaves
    (C) at least k leaves
    (D) exactly k leaves

19. A graph is said to be 2-colourable if each vertex can be coloured either red or blue and no two vertices of the same colour are connected by an edge. In a 2-colourable graph:

    (A) All cycles are of odd length.
    (B) There may be cycles of odd and even lengths.
    (C) All cycles are of even length.
    (D) The length of every cycle is divisible by 4.

20. In the programming language C, strings are represented as arrays of characters with a special character denoting the end of the string. For algorithms which reverse a C string, which of the following holds?

    (A) There is one without an auxiliary array taking time linear in the input.

(B) There is one without an auxiliary array taking constant time.

(C) All of them use an auxiliary array.

(D) All of them take time quadratic in the input.

21. The statements $x := x + y; y := x - y; x := x - y$ can be used for exchanging the value of two integer variables x and y except when:

    (A) one of x and y is negative

    (B) x and y are both negative and one of them has small absolute value

    (C) x, y both have the same sign and large absolute value

    (D) $|x|$ and $|y|$ are both small

22. The language $L = \{s \mid n(s) \textit{ is divisible by } 3\}$ where s is a string over the alphabet $\{0, 1\}$ representing a natural number $n(s)$ is:

    (A) regular

    (B) context-free but not regular

    (C) context-sensitive but not context-free

    (D) not context-sensitive

23. The reversal of a word w is denoted w^R (for example, $(word)^R = drow$). The reversal L^R of a language L is the set of reversals of its words. The palindromic reversal $L^P = \{w \in L \mid w = w^R\}$. $(L^P)^P$ is called a double palindromic reversal. Which of the following is true?

    (A) The set of all palindromes is a regular language.

    (B) Regular languages are closed under reversal.

    (C) Regular languages are closed under palindromic reversal.

    (D) Regular languages are closed under double palindromic reversal.

24. Given a language L, its prefix closure $Pref(L)$ is the set $\{p \mid p \textit{ is a prefix of } w \textit{ in } L\}$. Which of the following statements has a counterexample?

    (A) $Pref(L_1 \cup L_2) = Pref(L_1) \cup Pref(L_2)$

    (B) $Pref(L_1 \cap L_2) \subseteq Pref(L_1) \cap Pref(L_2)$

    (C) $Pref(L_1 L_2) = Pref(L_1) Pref(L_2)$

    (D) $Pref(\Sigma^*) = \Sigma^*$

25. Let $i = \sqrt{-1}$. For a prime p and a number m such that p does not divide m, the value of the sum $\Sigma_{k=0}^{p-1} e^{2\pi ikm/p}$ is:

    (A) 0 (B) 1 (C) e (the base of the natural logarithm) (D) $\pi^2/6$

PART B (6 questions, 40 marks)

1. A "five-pointed star" can be constructed by taking a regular pentagon and drawing straight lines from the first vertex to the third, the third to the fifth, the fifth to the second, and so on. Prove that the sum of the angles of the five vertices of the star is 180 degrees.

[4 Marks]

2. Given an array A of n integers, give an efficient algorithm to check whether there are two numbers in A whose sum is zero.

[5 Marks]

3. The "Towers of Hanoi" is a puzzle where there are three rods and n rings, all of different sizes, stacked on one of the rods, from smallest down to largest. The objective of the puzzle is to shift the entire stack to one of the other rods by moving one ring at a time, so that it is never the case that a larger ring is on top of a smaller one. Prove that this puzzle can be solved in $2^n - 1$ moves. For extra credit, show that this number is optimal.

[6 Marks]

4. A *triomino* is an L-shaped pattern made from three square tiles. A $2^k \times 2^k$ chessboard, whose squares are the same as tiles, has one arbitrary square painted red. Show that the chessboard can be covered with triominoes (as many as you need) leaving only the red square exposed.

[6 Marks]

5. An induced subgraph of a graph G contains a subset of vertices of the graph, and all the original edges between them. A vertex cover in a graph is a subset S of vertices such that every edge in the graph has at least one end point in S. An undirected graph G is *d-degenerate* (for some non negative integer d) if every induced subgraph of G has a vertex with degree at most d. A graph is said to be *d-colourable* if each vertex can be assigned one of d colours so that no two vertices of the same colour are connected by an edge.

   (a) Show that if a graph is d-degenerate then there is an ordering of the vertices of the graph into $v_1, v_2, ...v_n$ such that v_i has at most d edges to the vertices $v_1, ...v_{i-1}$.

   (b) Show that if a graph has a vertex cover S such that $|S| = d$, then the graph is d-degenerate.

   (c) Show that a d-degenerate graph is $d + 1$-colourable.

[9 Marks]

6. The function below takes a value *pivot* and an array $A[i..j]$ of integers and permutes the elements of A so that the ones below *pivot* are in the left half and the ones above *pivot* are in the right half.

```
function partition (var A: array of integer; i,j: unsigned; pivot: integer)
returns unsigned;
begin
L1:  while ((i < j) && (A[i] <= pivot)) do i := i+1;
L2:  while ((i < j) && (A[j] > pivot)) do j := j-1;
     if (A[j] > pivot) then j := j-1;
L3:  while (i < j) do begin
         t := A[i]; A[i] := A[j]; A[j] := t; i := i+1; j := j-1;
L4:      while (A[i] <= pivot) do i := i+1;
L5:      while (A[j] > pivot) do j := j-1
     end;
     return j
end
```

(a) Write the preconditions which should be assumed by a user of this function for it to work correctly.

(b) What does the return value of the function indicate to the calling program?

(c) Explain what goes wrong if for the boolean conditions in the first two loops (L1 and L2 respectively) we use the ones in the inner two loops (L4 and L5 resp.).

(d) Give a condition on the array A and the values i, j, n and *pivot* which is true when the loop L3 is entered and is again true when every iteration of the loop is completed, except after the last iteration when it is false. Give a short explanation of why your condition works in this "invariant" fashion.

[10 Marks]