(3 Hours) [Total Marks : 100

N. B. : (1) Question No. 1 is **compulsory**.
(2) Attempt any **four** questions from remaining **six** questions.
(3) **Figures** to **right** indicate **full** marks.

1. (a) Find Laplace Transform of $\int_0^t \frac{\sin u}{u}\, du$. 5

(b) Find the image of $| z - 3i | = 3$ under the mapping $w = \frac{1}{z}$. 5

(c) Obtain Laurent's series for $f(z) = \frac{4z + 3}{z(z-3)(z+2)}$ in the annular region between, $| z | = 2$ and $| z | = 3$. 5

(d) Show that sin x, sin 3x, sin 5x, ... form a set of orthogonal functions over $\left[0, \frac{\pi}{2}\right]$. Determine the corresponding orthonormal set. 5

2. (a) Show that $\int_0^\infty \frac{\cos \lambda x}{\lambda^2 + 1}\, d\lambda = \frac{\pi}{2} e^x,\ x \ge 0$. By definition of Fourier cosine integral. 6

(b) If f(z) is a regular function of z then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$. 6

(c) State and prove convolution theorem and hence find 8

$$L^{-1} \frac{(s+3)^2}{\left(s^2 + 6s + 5\right)^2}.$$

3. (a) Find Fourier series of 6

$$f(x) = \begin{cases} 0 & -2 \le x \le -1 \\ 1 + x & -1 \le x \le 0 \\ 1 - x & 0 \le x \le 1 \\ 0 & 1 \le x \le 2 \end{cases}$$

(b) Evaluate using Cauchy's Integral 6

Formula $\int_c \frac{z^2 + 4}{(z-2)(z+3i)}\, dz$, where c is

(i) $| z + 1 | = 2$
(ii) $| z - 2 | = 2$

(c) Use Laplace transform method to solve 8

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1, \text{ where } y(0) = 0,\ y'(0) = 1$$

4. (a) Using Cauchy s residue theorem evaluate

$$\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{z^2 + 3z + 2}\, dz, \text{ where c is (i) } |z| = 1, \quad \text{(ii) } |z| < 2.$$

(b) Find the analytic function $f(z) = u + iv$, If $v = e^x(x \cdot \sin y + y \cdot \cos y)$.

(c) Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that –

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}$$

5. (a) Find Inverse Laplace Transform of

(i) $\log\left(\frac{s+a}{s+b}\right)$ (ii) $\frac{8e^{-3s}}{s^2 + 4}$

(b) Using Laplace Transform, Evaluate $\int_0^\infty t^3 e^{-t} \sin dt$.

(c) Find the Bilinear transformation that maps the point $z = -i, 0, i$ into the points $w = -1, i, 1$ respectively. Into what curve the y-axis is transformed to this transformation.

6. (a) Find the complex form of Fourier series of $f(x) = e^{ax}$ $(-\pi < x < \pi)$ in the form

$$e^{ax} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{a + in}{a^2 + n^2} \cdot e^{inx}$$

(b) Show that the image of the rectangular hyperbola $x^2 - y^2 = 1$, under the transformation $w = \frac{1}{z}$ is the lemniscate.

(c) Evaluate –

(i) $\int_0^\pi \frac{d\theta}{3 + 2\cos\theta}$ (ii) $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$

7. (a) Find Laplace transform of

(i) $t^2 - e^{-2t} + \cosh^2 3t$ (ii) $e^t \cdot \sin 2t \cdot \sin 3t$

(b) If $f(a) = \int_c \frac{3z^2 + 7z + 1}{z - a}\, dz$

where c is a circle $|z| = 2$, then find –

(i) $f(-3)$ (ii) $f(i)$ (iii) $f(1 - i)$ (iv) $f'(1 - i)$

(c) Obtain the Fourier series for the function

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$$

Hence deduce that $\frac{\pi - 2}{4} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \text{-----------------}$