University of Hyderabad, ENTRANCE EXAMINATION
Ph.D. Mathematics/ Applied Mathematics

TIME: 2 hours **MAX. MARKS: 75**

Part A

Each question carries 1 mark. 1/4 mark will be deducted for each wrong answer. No deduction is made if the question is left unanswered. $\mathbb{R}$ is the set of real numbers, $\mathbb{Q}$ the set of rationals and $\mathbb{Z}$ the set of integers.

1. Let a_1, a_2 be positive real numbers and let $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for $n \geq 2$. Then the sequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$

(a) are not bounded. (b) both diverge to ∞. (c) both converge to the same limit. (d) both converge but to different limits.

2. Which of the following statements is true?

(a)The discontinuities of a monotonic function from $\mathbb{R}$ to $\mathbb{R}$ must be isolated.
(b)The discontinuities of a monotonic function from [0,1] to $\mathbb{R}$ must be isolated.
(c)The discontinuities of a monotonic function from $\mathbb{R}$ to $\mathbb{R}$ must be countable.
(d)The discontinuities of a monotonic function from $\mathbb{R}$ to $\mathbb{R}$ must be uncountable.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Which one of the following conditions implies that f is differentiable on $\mathbb{R}$?

(a) $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$ where c is a constant.
(b) $|f(x) - f(y)| \leq c|x - y|^{1/2}$ for all $x, y \in \mathbb{R}$ where c is a constant.
(c) f^2 is differentiable on $\mathbb{R}$.
(d) None of the above.

4. The number of Hausdorff topologies on a set with 100 elements is

(a) 1 (b) 2^{100} (c) 100^2 (d) 100!

5. Consider the following statements:

A: All Borel sets in $\mathbb{R}$ are Lebesgue measurable.
B: All Lebesgue measurable sets in $\mathbb{R}$ are Borel sets.
C: All Borel sets are either F_σ or G_δ sets.
(a) A, B, C are false. (b) only A is true. (c) only B is true. (d) only C is true.

6. Let $S = \mathbb{Q} \bigcup [-1,1] \bigcup (I \bigcap (2,3))$ where I is the set of irrational numbers. Then the Lebesgue integral $\int_{\mathbb{R}} \chi_S$ where χ_S is the indicator function of S is

(a) infinite. (b) 1. (c) 2. (d) 3.

7. The number of maximal ideals in $\frac{\mathbb{Z}}{36\mathbb{Z}}$ is

(a) 1. (b) 2. (c) 3. (d)4.

8. Let $A = \begin{bmatrix} M_{1,3} & 0 & 0 \\ 0 & M_{-1,2} & 0 \\ 0 & 0 & M_{-1,1} \end{bmatrix}$ where $M_{\lambda,n}$ is the $n \times n$ matrix $\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$.
Then the characteristic and minimal polynomials are given by

(a) $(X^2-1)^3$ and $(X^2-1)^2(X+1)$. (b) $(X-1)^4(X+1)^2$ and $(X^2-1)^2$. (c) $(X^2-1)^2(X-1)^2$ and $(X^2-1)^2(X+1)$. (d) $(X^2-1)^3$ and $(X^2-1)^2(X-1)$.

9. Which of the following is false:

(a) A subset of a metric space is compact if and only if it is closed and bounded. (b) In a compact metric space a family of closed sets with the property that any finite subfamily has nonempty intersection must itself have nonempty intersection. (c) A subset of a metric space is compact if and only if it is complete and totally bounded. (d) In a compact metric space every sequence has a convergent subsequence.

10. The Fourier series of $\sin^2 x + 2\cos^2 x$ on $[0,2\pi]$ is

(a) $\frac{3}{2} + \frac{1}{2}\cos 2x$. (b) $1 + \cos^2 x$. (c) $2\cos 2x + \sin 2x$. (d) none of the above.

11. Let $||x||_1 = \sum_{i=1}^{n} |x_i|$ and $||x||_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ induce topologies τ_1 and τ_2 on $\mathbb{R}^n$; then

(a) τ_1 is weaker than τ_2. (b) τ_1 is stronger than τ_2. (c) τ_1 is equivalent to τ_2. (d) τ_1 and τ_2 are not comparable.

12. As a subset of [0,1] equipped with the usual topology the cantor set is

(a) nowhere dense, uncountable but not compact. (b) closed, dense and uncountable. (c) not closed, dense and countable. (d) compact, nowhere dense and uncountable.

13. Consider the statements:
A: all real symmetric $n \times n$ matrices are diagonalizable over $\mathbb{R}$.
B: all complex $n \times n$ matrices are digonalizable.
C: all nilpotent $n \times n$ matrices are diagonalizable.

(a) all three statements are false. (b) A and B alone are false. (c) B and C alone are false. (d) A and C alone are false.

14. The total number of nontrivial proper subgroups of a cyclic group of order 24 is

(a) 2 (b) 4 (c) 6 (d) 8.

15. The unique natural number less than 36 to which 2^{12^7+3} is congruent modulo 36 is

(a) 8 (b) 6 (c) 4 (d) 2.

16. The number of proper subfields of the finite field with 3^6 elements is

(a) 1 (b) 2 (c) 3 (d) 6.

17. The number of possible Jordan canonical forms with characteristic polynomial $(X-2)^7$ and minimal polynomial $(X-2)^3$ is
(a) 5 (b) 4 (c) 3 (d) 2.

18. The particular solution of $\frac{d^2y}{dx^2} + y = \csc x$ is given by

(a)$\sin x \log(\sin x) - x \cos x$ (b) $\sin x \log(\sin x) + x \cos x$ (c) $(\sin x)(\log x) - x \sin x$ (d) $\sin x \log(\cos x) + x \sin x$

19. How many solutions are there for the partial differential equation $z^2 = p^2 - q^2$ passing through the curve $x_0 = s, y_0 = 0, z_0 = -s^2/4$?

(a) No solution exists (b) There is a unique solution. (c) Two solutions (d) More than two solutions.

20. The partial differential equation $y^3 u_{xx} - (x-1)^2 u_{yy} = 0$ is

(a) hyperbolic in $\{(x, y)|\ y < 0\}$. (b) hyperbolic in $\{(x, y)|y > 0\}$ (c) elliptic in $\{(x, y)|y > 0\}$. (d) parabolic in $\mathbb{R}^2$.

21. The statement " Every nonzero normed linear space admits a nonzero bounded linear functional" follows from

(a) The Hahn Banach theorem (b) The uniform boundedness principle (c) The open mapping theorem (d) The closed graph theorem.

22. The number of zeroes of $f(z) = 2z^5 + 6z - 1$ in the annulus $1 < |z| < 2$ is

(a) 5 (b) 4 (c) 3 (d) 1.

23. $\cot iz$ is equal to

(a) $\coth z$ (b) $-\coth z$ (c) $i \coth z$ (d) $-i \coth z$.

24. The function $f(z) = \frac{e^z}{z(1-e^{-z})}$, the point $z = 0$ is

(a) a removable singularity (b) an essential singularity (c) a pole of order 1 (d) a pole of order 2.

25. The critical point (0,0) for the system $\frac{dx}{dt} = x - y,\ \ \frac{dy}{dt} = x + 5y$ is

(a) a stable node (b) an unstable node (c) a saddle point (d) a spiral.

Part B

Each question carries 5 marks. Answer any 10 questions.

1. Prove that a group of order 77 must be cyclic. (Do not just quote a result).

2. If F_n denotes the finite field with n elements show that the Galois group $G(F_{5^{12}}/F_{5^2})$ is cyclic. What is its order?

3. Write down all possible Jordan canonical forms for a 7×7 matrix for which the minimal polynomial is $X^3(X-1)^2$.

4. Determine an orthogonal matrix A such that $A^T BA$ is diagonal where $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

5. Give an example of a unique factorization domain that is not a principal ideal domain.

6. Evaluate by using the residue theorem $\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\, d\theta$.

7. Find a one-to-one holomorphic function from the upper half plane $\{z \mid \text{Im} z > 0\}$ in $\mathbb{C}$ to the open unit disc.

8. Find the nature of the extremum for the functional $J(y) = \int_1^2 \frac{x^3}{y'^2}\, dx$; $y(1) = 1,\ y(2) = 4$.

9. Construct the Green's function for the boundary value problem $x^2y'' + xy' - y = 0, \quad 1 \le x \le 2$ subject to the boundary conditions $y'(1) + y'(2) = 0$ and $y(1) = 0$.

10. Find the characteristic strips of $pq = z$ and hence find the integral surface passing through the parabola $x = 0, \quad y^2 = z$.

11. Investigate the solvability of the integral equation
$\phi(x) - \lambda \int_0^\pi \cos^2 x \ \phi(t) \ \ dt = 1.$

12. Consider $\mathbb{R}$ with the cofinite topology τ, i.e., the closed sets are $\mathbb{R}$ and all finite subsets. Is $(\mathbb{R}, \tau)$ metrizable? Why? Show that $(\mathbb{R}, \tau)$ is compact and connected.

13. State Lebesgue's dominated convergence theorem and show by an example that the sequence of functions must be dominated by an integrable function. Can the integrable function be a constant function?

14. Chose $x_1 > 1$ and define $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{x_n})$. Discuss the convergence of the sequence $\{x_n\}$.

15. Let $\{x_n | n = 1, 2, 3, ...\}$ be a countable number of points in $\mathbb{R}$. Construct a function which is discontinuous at exactly the points x_n and nowhere else.