2009

Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is

correct.

3. Marking scheme: Each correct answer will be awarded 2:5 marks, but 0:5 marks

will be deducted for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the ques-

tion will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely be-

fore marking the other bubble.

9. Let N, Z, Q, R and C denote the set of natural numbers, the set of integers, the

set of rational numbers, the set of real numbers and the set of complex numbers

respectively.

10. Let [x] denote the greatest integer less than or equal to x for a real number x.

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Integrated Ph. D./ Mathematical Sciences

1. Let T and S be linear transformations from R2 to R2. Let T rotate each point

counterclockwise through an angle \_ about the origin and let S be the reection

about the line y = x. Then determinant of TS is

(A) 1.

(B) 􀀀1.

(C) 0.

(D) 2.

2. Let V be a 7 dimensional vector space. Let W and Z be subspaces of V with

dimensions 4 and 5 respectively. Which of the following is not a possible value of

dim(W \ Z) ?

(A) 1.

(B) 2.

(C) 3.

(D) 4.

3. If a; b 2 R satisfy a2 + 2ab + 2b2 = 7, then the largest possible value of ja 􀀀 bj is

(A) p7.

(B) r7

2

.

(C) p35.

(D) 7.

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4. Suppose a \_nite group G has an element a which is not the identity such that a20

is the identity. Which of the following can not be a possible value for the number

of elements of G?

(A) 12.

(B) 9.

(C) 20.

(D) 15.

5. Let A be a 10 \_ 10 matrix in which each row has exactly one entry equal to 1,

the remaining nine entries of the row being 0. Which of the following is not a

possible value for the determinant of the matrix A?

(A) 0.

(B) 􀀀1.

(C) 10.

(D) 1.

6. A subset V of R3 consisting of vectors (x1; x2; x3) satisfying x2

1 + x2

2 + x2

3 = k is a

subspace of R3 if k is

(A) 0.

(B) 1.

(C) 􀀀1.

(D) none of the above.

7. Let v1 = (1; 0); v2 = (1;1) and v3 = (0; 1). How many linear transformations

T : R2 ! R2 are there such that Tv1 = v2; Tv2 = v3 and Tv3 = v1?

(A) 3!.

(B) 3.

(C) 1.

(D) 0.

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8. The equation x3 + 7x2 + 1 + ixe􀀀x = 0 has

(A) no real solution.

(B) exactly one real solution.

(C) exactly two real solutions.

(D) exactly three real solutions.

9. How many complex numbers z = x + iy are there such that x + y = 1 and

exp i(x2 + y2) = 1?

(A) Zero.

(B) Non-zero but \_nitely many.

(C) Countably in\_nite.

(D) Uncountably in\_nite.

10. Let G = fg1; g2; : : : ; gng be a \_nite group and suppose it is given that g2

i = identity

for i = 1; 2; : : : ; n 􀀀 1. Then

(A) g2

n is identity and G is abelian.

(B) g2

n is identity, but G could be non-abelian.

(C) g2

n may not be identity.

(D) none of the above can be concluded from the given data.

11. Let X = f2; 3; 4; : : : g be the set of integers greater than or equal to 2. Consider

the binary relation R on X given by the following: mRn if m and n have a

common integer factor r 6= 1. Then R is

(A) reexive and transitive but not symmetric.

(B) reexive and symmetric but not transitive.

(C) symmetric and transitive but not reexive.

(D) an equivalence relation.

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12. If X and Y are two non-empty \_nite sets and f : X ! Y and g : Y ! X are

mappings such that g \_ f : X ! X is a surjective (i.e., onto) map, then

(A) f must be one-to-one.

(B) f must be onto.

(C) g must be one-to-one.

(D) X and Y must have the same number of elements.

13. Let X and Y be two non-empty sets and let f : X ! Y , g : Y ! X be two

mappings. If both f and g are injective (i.e., one-to-one) then

(A) X and Y must be in\_nite sets.

(B) g = f􀀀1 always.

(C) one of f \_ g : Y ! X and g \_ f : X ! Y is always bijective (one-to-one and

onto).

(D) There exists a bijective mapping h: X ! Y .

14. Consider the system of linear equations

a1x + b1y + c1z = d1;

a2x + b2y + c2z = d2;

a3x + b3y + c3z = d3;

where ai; bi; ci; di are real numbers for 1 \_ i \_ 3. If\_\_\_\_\_\_

b1 c1 d1

b2 c2 d2

b3 c3 d3

\_\_\_\_\_\_

6= 0 then the

above system has

(A) at most one solution.

(B) always exactly one solution.

(C) more than one but \_nitely many solutions.

(D) in\_nitely many solutions.

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15. Consider the group G = Z4 \_ Z4 of order 16, where the operation is component

wise addition modulo 4. If G is a union of n subgroups of order 4 then the

minimum value of n is

(A) 4.

(B) 5.

(C) 6.

(D) 7.

16. The altitude of a triangle is a line which passes through a vertex of the triangle and

is perpendicular to the opposite side. The orthocenter is the point of intersection

of the three altitudes. Let A be the triangle whose vertices are (1; 0), (3;1) and

(0; 3). Then the orthocenter of A is

(A) (4=3; 2=3).

(B) (􀀀3;3).

(C) (􀀀1; 1).

(D) (3; 5).

17. The area of the triangle formed by the straight lines 8x 􀀀 3y = 48, 7y + 4x = 24

and 5y 􀀀 2x = 22 is

(A) 26.

(B) 30.

(C) 34.

(D) 36.

18. The equation x2 􀀀 y2 + (a + b)x + (a 􀀀 b)y + c = 0 represents

(A) either a hyperbola or a pair of straight lines.

(B) always a hyperbola.

(C) always a pair of straight lines.

(D) always a parabola.

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19. If the volume of the tetrahedron whose vertices are (1; 1; 1), (3; 2; 0), (0; 4; 3) and

(5; 0; k) is 6 then the value of k is

(A) 􀀀16=7.

(B) 􀀀4=7.

(C) 2=7.

(D) 2.

20. Which one of the following curves intersects every plane in the 3-dimensional

Euclidean space R3 ?

(A) (x; y; z) = (t; t2; t3).

(B) (x; y; z) = (t; t3; t4).

(C) (x; y; z) = (t; t3; t5).

(D) (x; y; z) = (t; t2; t5).

21. Let Q = (0; 0; b) and R = (0; 0;b) be two points in the 3-dimensional Euclidean

space R3. If the di\_erence of the distances of a point P in R3 from Q and R is

2a (where a 6= \_b) then the locus of P is

(A)

x2

b2 􀀀 a2 +

y2

b2 􀀀 a2 􀀀

z2

a2 􀀀 1 = 0.

(B)

x2

b2 􀀀 a2 +

y2

b2 􀀀 a2 􀀀

z2

a2 + 1 = 0.

(C)

x2

a2 􀀀 b2 +

y2

a2 􀀀 b2 􀀀

z2

a2 + 1 = 0.

(D)

x2

b2 +

y2

b2 􀀀

z2

a2 + 1 = 0.

22. De\_ne a function f on the real line by

f(x) = \_ x 􀀀 [x] 􀀀 1

2 if x is not an integer;

0 if x is an integer

Then which of the following is true:

(A) f is periodic with period 1, i.e., f(x + 1) = f(x) for all x.

(B) f is continuous.

(C) f is one-to-one.

(D) limx!a f(x) exists for all a 2 R.

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23. Let a; b and c be non-zero real numbers. Let

f(x) = \_ sin x if x \_ c

ax + b if x > c

Suppose b and c are given. Then

(A) There is no value of a for which f is continuous at c.

(B) There is exactly one value of a for which f is continuous at c.

(C) There are in\_nitely many values of a for which f is continuous at c.

(D) Continuity of f at c can not be determined from what is given.

24. Let

f(x) = \_ 1 if jxj \_ 1;

0 if jxj > 1

and g(x) = 2 􀀀 x2:

Let h(x) = f(g(x)). Then h(x)

(A) is continuous everywhere.

(B) has exactly one point of discontinuity.

(C) has exactly two points of discontinuity.

(D) has four points of discontinuity.

25. Let 0 < a < b. De\_ne a function M(r) for a \_ r \_ b by

M(r) = maxf

r

a 􀀀 1; 1 􀀀

r

b g:

Then minfM(r) : a \_ r \_ bg is

(A) 0.

(B) 2ab=(a + b).

(C) (b 􀀀 a)=(b + a).

(D) (b + a)=(b 􀀀 a).

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26. Let f : R ! R be a function such that f(x + y) = f(x)f(y) for all x; y 2 R and

f(x) = 1 + xg(x) where limx!0 g(x) = 1. Then the function f(x) is

(A) ex,

(B) 2x,

(C) a non-constant polynomial,

(D) equal to 1 for all x 2 R.

27. Let f(x) be a continuous function on [0; a] such that f(x)f(a 􀀀 x) = 1. Then

Z a

0

dx

1 + f(x)

is

(A) 0,

(B) 1,

(C) a,

(D) a=2.

28. Let f : [0;1) ! [0;1) satisfy

(f(x))2 = 1 + 2 Z x

0

f(t)dt:

Then f(1) is

(A) loge 2,

(B) 1,

(C) 2,

(D) e.

29. Let

f(x) = Z x

1

et

t

dt

for x \_ 1. Then f(x) > loge x

(A) for no value of x.

(B) only for x > e.

(C) for 1 \_ x \_ e.

(D) for all x > 1.

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30. Consider the \_rst order ODE

dy

dx

= F \_ ax + by + c

Ax + By + C\_

where a; b; c; A;B and C are non-zero constants. Under what condition, does

there exist a linear substitution that reduces the equation to one in which the

variables are separable?

(A) Never.

(B) if aB = bA.

(C) if bC = cB.

(D) if cA = aC.

31. Let ' be a solution of the ODE

x2y0 + 2xy = 1 on 0 < x < 1:

Then the limit of '(x) as x ! 1

(A) is zero.

(B) is one.

(C) is 1.

(D) does not exist.

32. Let ' be the solution of y0 + iy = x such that '(0) = 2. Then '(\_) equals

(A) i\_.

(B) 􀀀i\_.

(C) \_.

(D) 􀀀\_.

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33. Consider the matrix

A = \_a

c

b

d\_

with real entries. Suppose it has repeated eigenvalues. Pick the correct statement:

(A) bc = 0.

(B) A is always a diagonal matrix.

(C) det(A) \_ 0.

(D) det(A) can take any real value.

34. Let G denote the group of all 2\_2 real matrices with non-zero determinant. Let

H denote the subgroup of all matrices with determinant 1. Let G=H denote the

set of left cosets of H. Then

(A) H is not a normal subgroup.

(B) G=H is isomorphic to the real numbers under addition.

(C) G=H is isomorphic to the non-zero real numbers under multiplication.

(D) G=H is a \_nite group.

35. Let ~a and ~b be two non-zero vectors in R3 such that j~a \_~bj = ~a:~b. Then the

smaller of the two angles subtended by ~a and ~b is

(A) zero.

(B) an acute angle.

(C) a right angle.

(D) an obtuse angle.

36. Let f : R ! R be a di\_erentiable function such that f0 is continuous. De\_ne the

function

G(x; y) = f(px2 + y2) for all (x; y) 2 R2:

Then

(A) @G

@x and @G

@y are always continuous at each (x; y) 2 R2.

(B) @G

@x and @G

@y always exist but are not continuous at some point.

(C) G is always continuous on R2.

(D) The continuity of G depends on the choice of f.

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37. The value of the integral Z 1

0 Z 1

y

yp1 + x3dxdy

is

(A) 2p2.

(B) (2p2 􀀀 1)=2.

(C) (2p2 􀀀 1)=8.

(D) (2p2 􀀀 1)=9.

38. Consider the pair of \_rst order ordinary di\_erential equations

dx

dt

= Ax + By;

dy

dt

= x;

where B < 􀀀1 < A < 0. Let (x(t); y(t)) be the solution of the above that satis\_es

(x(0); y(0)) = (0; 1). Pick the correct statement:

(A) (x(t); y(t)) = (0; 1) for all t 2 R.

(B) x(t) is bounded on R.

(C) x(t) is bounded on [0;1).

(D) y(t) is bounded on R.

39. Let f(x) be a non-constant second degree polynomial such that f(2) = f(􀀀2).

If the real numbers a; b and c are in arithmetic progression, then f0(a); f0(b) and

f0(c) are

(A) in arithmetic progression.

(B) in geometric progression.

(C) in harmonic progression.

(D) equal.

40. Let P(x) be a non-constant polynomial such that P(n) = P(􀀀n) for all n 2 N.

Then P0(0)

(A) equals 1.

(B) equals 0.

(C) equals 􀀀1.

(D) can not be determined from the given data.

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