M.Math. Sample Questions 2005

1. Let $F : \mathbf{R}^n \to \mathbf{R}$ be defined by

$$F(x_1, x_2, ..x_n) = \max\{| x_1 |, | x_2 |, ... | x_n |\}.$$

Show that F is a uniformly continuous function.

2. Let $f : (0,1) \to \mathbf{R}$ be defined as
$f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ with m, n relatively prime integers and $f(x) = 0$ if x is irrational.
Let $g : \mathbf{R} \to \mathbf{R}$ be defined as
$g(x) = 0$ if $x \leq 0$ or $x > \frac{1}{2}$ and $g(x) = 1$ for other x.
Show that the function $h(x) = g(f(x))$ is not Riemann integrable.

3. A map $f : \mathbf{R} \to \mathbf{R}$ is called open if $f(A)$ is open for every open subset A of $\mathbf{R}$. Show that every continuous open map of $\mathbf{R}$ into itself is monotonic.

4. Let (X, d) be a compact metric space. Show that every map $f : X \to X$ satisfying $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$ is onto.

5. Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ and $f : [0,1] \to \mathbf{C}$ be continuous with $f(0) = 0, \ f(1) = 2$. Show that there exists at least one t_0 in $[0,1]$ such that $f(t_0)$ is in $\mathbf{T}$.

6. Let f be a continuous function on $[0,1]$. Evaluate

$$\lim_{n \to \infty} \int_0^1 x^n f(x) dx$$

7. Find the most general curve whose normal at each point passes though $(0,0)$. Find the particular curve through $(2,3)$.

8. Suppose f is a contnuous function on $\mathbf{R}$ which is periodic with period 1, that is, $f(x+1) = f(x)$ for all x. Show that
   (i) the function f is bounded above and below,
   (ii) it achieves both its maximum and minimum and
   (iii) that it is uniformly continuous.

9. Let $S = \{(x_1, x_2, ...x_n) \in \mathbf{R}^n : \sum \mid x_i \mid^2 = 1\}$. Let

$$A = \{(y_1, y_2, ...y_n) \in \mathbf{R}^n : \sum \frac{y_i}{i} = 0\}.$$

Show that the set $S + A = \{x + y : x \in S \ , \ y \in A\}$ is a closed subset of $\mathbf{R}^n$.

10. Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \geq j$. Prove that A^n is the zero matrix.

11. Determine the integers n for which Z_n, the set of integers modulo n, contains elements x, y so that $x + y = 2, 2x - 3y = 3$.

12. Let a_1, b_1 be arbitrary positive real numbers. Define

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$

for all $n \geq 1$. Show that a_n and b_n converge to a common limit.

13. Show that the only field automorphism of $\mathbf{Q}$ is the identity. Using this prove that the only field automorphism of $\mathbf{R}$ is the identity.

14. Consider a circle which is tangent to the y-axis at 0. Show that the slope at any point (x, y) satisfies $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.

15. Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0 \ \forall \ (i, j) \neq (1, 2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.

16. Let G be a nonabelian group of order 39. How many subgroups of order 3 does it have?

17. Let $n \in \mathbf{N}$, let p be a prime number and let $\mathbf{Z}_{p^n}$ denote the ring of integers modulo p^n under addition and multiplication modulo p^n. Let $f(x) = \sum_{i=1}^{r} a_i x^i$ and $g(x) = \sum_{i=1}^{s} b_i x^i$ be polynomials with coefficients from the ring $\mathbf{Z}_{p^n}$ which satisfy $f(x) \cdot g(x) = 0$. Prove that $a_i b_j = 0 \ \forall \ i, j$.

18. Show that the fields $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ are isomorphic as $\mathbf{Q}$-vector spaces but not as fields.

19. Prove that $X^4 - 10X^2 + 1$ is reducible modulo p for every prime p.

20. Suppose $a_n \geq 0$ and that $\sum a_n$ is convergent. Show that $\sum 1/(n^2 a_n)$ is divergent.