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2008-MA

Test Paper Code: MA

Time: 3 Hours Maximum Marks: 300

INSTRUCTIONS

1. The question-cum-answer booklet has **36** pages and has **29** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Roll Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **5**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used:
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **-2 (Negative two)** marks.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone, pager and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

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2008-MA

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER						

Name:

Test Centre:

Do not write your Roll Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

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Signature of the Invigilator

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Special Instructions/ Useful Data

$\mathbb{Z}$	- Set of all integers
$\mathbb{Q}$	- Set of all rational numbers
$\mathbb{R}$	- Set of all real numbers
$\mathbb{C}$	- Set of all complex numbers
A^T	- Transpose of the matrix A
$\overline{A}^T$	- Conjugate transpose of the matrix A
det A	- Determinant of the matrix A

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IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-29 (subjective questions) carry *fifteen* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 5 only.**

Q.1 The least positive integer n, such that $\begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}^n$ is the identity matrix of order 2, is

(A) 4 (B) 8 (C) 12 (D) 16

Q.2 Let $S = \{T : \mathbb{R}^3 \to \mathbb{R}^3 \mid T$ is a linear transformation with $T(1,0,1) = (1,2,3)$, $T(1,2,3) = (1,0,1)\}$. Then S is

(A) a singleton set (B) a finite set containing more than one element
(C) a countably infinite set (D) an uncountable set

Q.3 Let $s_n = \int_0^1 \frac{n x^{n-1}}{(1+x)}\,dx$ for $n \geq 1$. Then as $n \to \infty$, the sequence $\{s_n\}$ tends to

(A) 0 (B) $1/2$ (C) 1 (D) $+\infty$

Q.4 The work done by the force $\vec{F} = 4y\,\hat{i} - 3xy\,\hat{j} + z^2\,\hat{k}$ in moving a particle over the circular path $x^2 + y^2 = 1$, $z = 0$ from $(1,0,0)$ to $(0,1,0)$ is

(A) $\pi + 1$ (B) $\pi - 1$ (C) $-\pi + 1$ (D) $-\pi - 1$

Q.5 The set of all boundary points of $\mathbb{Q}$ in $\mathbb{R}$ is

(A) $\mathbb{R}$ (B) $\mathbb{R} \setminus \mathbb{Q}$ (C) $\mathbb{Q}$ (D) $\varnothing$

Q.6 Let $V = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{1}{4} \leq x^2 + y^2 + z^2 \leq 1\right\}$ and $\vec{F} = \frac{x\,\hat{i} + y\,\hat{j} + z\,\hat{k}}{(x^2+y^2+z^2)^2}$ for $(x,y,z) \in V$. Let $\hat{n}$ denote the outward unit normal vector to the boundary of V and S denote the part $\left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = \frac{1}{4}\right\}$ of the boundary of V. Then $\iint_S \vec{F} \cdot \hat{n}\, dS$ is equal to

(A) -8π (B) -4π (C) 4π (D) 8π

Q.7 The set $U=\left\{x \in \mathbb{R} \mid \sin x = \frac{1}{2}\right\}$ is

(A) open
(B) closed
(C) both open and closed
(D) neither open nor closed

Q.8 Let $f(x)=\int_0^x (x^2 - t^2)\, g(t)\, dt$, where g is a real valued continuous function on $\mathbb{R}$. Then $f'(x)$ is equal to

(A) 0
(B) $x^3\, g(x)$
(C) $\int_0^x g(t)\, dt$
(D) $2x \int_0^x g(t)\, dt$

Q.9 Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation $y'' + P(x)y' + Q(x)y = 0$, where $P(x)$ and $Q(x)$ are continuous functions on an interval I. Then $y_3(x) = a\,y_1(x) + b\,y_2(x)$ and $y_4(x) = c\,y_1(x) + d\,y_2(x)$ are linearly independent solutions of the given differential equation if

(A) $ad = bc$
(B) $ac = bd$
(C) $ad \neq bc$
(D) $ac \neq bd$

Q.10 The set $R=\left\{f \mid f \text{ is a function from } \mathbb{Z} \text{ to } \mathbb{R}\right\}$ under the binary operations $+$ and $\cdot$ defined as $(f+g)(n) = f(n) + g(n)$ and $(f \cdot g)(n) = f(n)\, g(n)$ for all $n \in \mathbb{Z}$ forms a ring. Let $S_1 = \left\{f \in R \mid f(-n) = f(n) \text{ for all } n \in \mathbb{Z}\right\}$ and $S_2 = \left\{f \in R \mid f(0) = 0\right\}$. Then

(A) S_1 and S_2 are both ideals in R
(B) S_1 is an ideal in R while S_2 is not
(C) S_2 is an ideal in R while S_1 is not
(D) neither S_1 nor S_2 is an ideal in R

Q.11 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,2,3) = (1,2,3)$, $T(1,5,0) = (2,10,0)$ and $T(-1,2,-1) = (-3,6,-3)$. The dimension of the vector space spanned by all the eigenvectors of T is

(A) 0
(B) 1
(C) 2
(D) 3

Q.12 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers defined as $a_1 = 1$ and for $n \geq 1$, $a_{n+1} = a_n + (-1)^n 2^{-n}$, $b_n = \dfrac{2a_{n+1} - a_n}{a_n}$. Then

(A) $\{a_n\}$ converges to zero and $\{b_n\}$ is a Cauchy sequence
(B) $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is a Cauchy sequence
(C) $\{a_n\}$ converges to zero and $\{b_n\}$ is not a convergent sequence
(D) $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is not a convergent sequence

Q.13 Let $f : (-1,1) \to \mathbb{R}$ be defined as $f(x) = \dfrac{x^2}{1 - \cos x}$ for $x \neq 0$ and $f(0) = 2$. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the Taylor expansion of f for all x in $(-1,1)$, then $\sum_{n=0}^{\infty} a_{2n+1}$ is

(A) 0 (B) $1/2$ (C) 1 (D) 2

Q.14 Let $y_1(x)$ and $y_2(x)$ be twice differentiable functions on an interval I satisfying the differential equations $\dfrac{dy_1}{dx} - y_1 - y_2 = e^x$ and $2\dfrac{dy_1}{dx} + \dfrac{dy_2}{dx} - 6y_1 = 0$. Then $y_1(x)$ is

(A) $C_1 e^{-2x} + C_2 e^{3x} - \frac{1}{4} e^x$
(B) $C_1 e^{2x} + C_2 e^{-3x} + \frac{1}{4} e^x$
(C) $C_1 e^{2x} + C_2 e^{-3x} - \frac{1}{4} e^x$
(D) $C_1 e^{-2x} + C_2 e^{3x} + \frac{1}{4} e^x$

Q.15 Let G be a finite group and H be a normal subgroup of G of order 2. Then the order of the center of G is

(A) 0 (B) 1 (C) an even integer ≥ 2 (D) an odd integer ≥ 3

Space for rough work

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Space for rough work

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Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question No. Do not write anything else on this page.

Question No.	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

No. of Correct Answers		Marks	(+)
No. of Incorrect Answers		Marks	(–)
Total Marks in Question Nos. 1-15			( )

A

Q.16 (a) Let f and g be continuous functions on $\mathbb{R}$ such that $f(x)=\int_0^x g(t)\,dt$ and $g(x)=\int_x^0 f(t)\,dt+1$. Prove that $(f(x))^2+(g(x))^2=1$ for all $x\in\mathbb{R}$. (6)

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that f' is continuous on $\mathbb{R}$. Show that the series $\sum_{n=1}^{\infty}\left(f\left(\frac{x}{2n}\right)-f\left(\frac{x}{2n+1}\right)\right)$ converges uniformly on $[0,1]$. (9)

A

Q.17 (a) Find the maxima, minima and saddle points, if any, for the function $f(x,y) = (y - x^2)(y - 2x^2)$ on $\mathbb{R}^2$. (6)

(b) Let $P(x) = a_0 + a_1 x^2 + a_2 x^4 + \cdots + a_n x^{2n}$, where $n \geq 1$ and $a_k > 0$ for $k = 0$, $1, \cdots, n$. Show that $P(x) - x P'(x) = 0$ has exactly two real roots. (9)

A

A

Q.18 (a) Given that $y_1(x) = x$ is a solution of $(1+x^2)y'' - 2xy' + 2y = 0$, $x > 0$, find a second linearly independent solution. (6)

(b) Solve $x^2y'' + xy' - y = 4x \log x$, $x > 0$. (9)

A

Q.19 (a) Let ϕ be a differentiable function on $[0,1]$ satisfying $\phi'(x) \leq 1+3\phi(x)$ for all $x \in [0,1]$ with $\phi(0)=0$. Show that $3\phi(x) \leq e^{3x}-1$. (6)

(b) If $y_1(x)=x(1-2x)$, $y_2(x)=2x(1-x)$ and $y_3(x)=x(e^x-2x)$ are three solutions of a non-homogeneous linear differential equation $y''+P(x)y'+Q(x)y=R(x)$, where $P(x)$, $Q(x)$ and $R(x)$ are continuous functions on $[a,b]$ with $a>0$, then find its general solution. (9)

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Q.20 (a) Evaluate $\int_{1}^{4}\int_{0}^{1}\int_{2y}^{2} \frac{\cos x^2}{\sqrt{z}}\, dx\, dy\, dz$. (6)

(b) Find the surface area of the portion of the cone $z^2 = x^2 + y^2$ that is inside the cylinder $z^2 = 2y$. (9)

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A

Q.21 (a) Use Green's theorem to evaluate the integral $\oint_C x^2\,dx + (x + y^2)\,dy$, where C is the closed curve given by $y = 0$, $y = x$ and $y^2 = 2 - x$ in the first quadrant, oriented counter clockwise. (6)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Use change of variables to prove that $\iint_D f(x-y)\,dx\,dy = \int_{-1}^{1} f(u)\,du$, where $D = \left\{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\right\}$. (9)

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Q.22 Using Gauss's divergence theorem, evaluate the integral $\iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + 4yz\,\hat{k}$, S is the surface of the solid bounded by the sphere $x^2 + y^2 + z^2 = 10$ and the paraboloid $x^2 + y^2 = z - 2$, and $\hat{n}$ is the outward unit normal vector to S. (15)

A

Q.23 (a) A square matrix M of order n with complex entries is called skew Hermitian if $M+\overline{M}^T=0$, where 0 is the zero matrix of order n.

Determine whether $V=\{M \mid M \text{ is a } 2\times 2 \text{ skew Hermitian matrix }\}$ is a vector space over (i) the field $\mathbb{R}$ and (ii) the field $\mathbb{C}$ with the usual operations of addition and scalar multiplication for matrices? (6)

(b) Let $V=\{P(x) \mid P(x) \text{ is a polynomial of degree } \leq n \text{ with real coefficients }\}$ and $T:V\rightarrow\mathbb{R}^m$ be defined as $T(P(x))=(P(1),P(2),\cdots,P(m))$. Then show that T is linear and determine the Nullity of T. (9)

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Q.24 Let G be the set of all 3×3 real matrices M such that $MM^T = M^T M = I_3$ and let $H = \{M \in G \mid \det M = 1\}$, where I_3 is the identity matrix of order 3. Then show that

(i) G is a group under matrix multiplication,

(ii) H is a normal subgroup of G,

(iii) $\phi : G \to \{-1, 1\}$ given by $\phi(M) = \det M$ is onto,

(iv) G / H is abelian. (15)

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Q.25 (a) Suppose that $(R,+,\cdot)$ is a ring having the property $a\cdot b=c\cdot a \Rightarrow b=c$, when $a \neq 0$. Then prove that $(R,+,\cdot)$ is a commutative ring. (6)

(b) Let R be a commutative ring with identity. For $a_1, a_2, \cdots, a_n \in R$, the ideal generated by $\{a_1, a_2, \cdots, a_n\}$ is given by

$$\langle a_1, a_2, \cdots, a_n \rangle = \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \mid r_i \in R, 1 \leq i \leq n\}.$$

Let $\mathbb{Z}[x]$ be the set of all polynomials with integer coefficients. Consider the ideal $I = \{f \in \mathbb{Z}[x] \mid f(0) \text{ is an even integer}\}$. Prove that $I = \langle 2, x \rangle$ and that it is a maximal ideal. (9)

Q.26 For a given positive integer $n > 1$, show that there exist subspaces $X_1, X_2, \cdots, X_n$ of $\mathbb{R}^m$ for some integer $m > n$ and a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ such that

- $\dim X_k = k$, $k = 1, 2, \cdots, n$,
- for $i \neq j$, $X_i \cap X_j = \{\vec{0}\}$, where $\vec{0}$ is the zero vector of $\mathbb{R}^m$,
- $T(X_k) = X_{k-1}$, $k = 1, 2, \cdots, n$, where $X_0 = \{\vec{0}\}$.

Also, find the rank of T. (15)

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Q.27 Let $f:(0,\infty)\to(0,\infty)$ be a continuously differentiable function and let $z=\dfrac{xy}{f(x^2+y^2)}$ be defined for $xy\neq 0$.

(a) Prove that $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=\dfrac{(x+y)}{[f(x^2+y^2)]^2}\left\{f(x^2+y^2)-2xy\,f'(x^2+y^2)\right\}$. (6)

(b) Further, if f is homogeneous of degree $\dfrac{1}{2}$, then verify that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$. (9)

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Q.28 Determine the interval of convergence of the power series $\sum_{n=1}^{\infty} n(2n-1)x^{2n}$ and show that its sum is $\dfrac{x^2(1+3x^2)}{\left(1-x^2\right)^3}$ at any point x in its interval of convergence.

(15)

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Q.29 (a) Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x,y) = x^2\cos(y/x)$ for $x \neq 0$ and $f(x,y) = 0$ for $x = 0$. Compute $\dfrac{\partial f}{\partial x}$ at all points in $\mathbb{R}^2$ and show that it is continuous at the origin. (6)

(b) Let $f:(0,1) \to (0,\infty)$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in $(0,1)$, then prove that $\{f(x_n)\}$ is a Cauchy sequence in $(0,\infty)$. Hence deduce that for any two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $(0,1)$, $\{|f(x_n) - f(y_n)|\}$ is a Cauchy sequence in $(0,\infty)$. (9)

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