Test Code MS (Short answer type) 2004

Syllabus for Mathematics

Permutations and Combinations. Binomials and multinomial theorem. Theory of equations. Inequalities.

Determinants, matrices, solution of linear equations and vector spaces.

Trigonometry, Coordinate geometry of two and three dimensions.

Geometry of complex numbers and De Moivre's theorem. Elements of set theory.

Convergence of sequences and series. Power series. Functions, limits and continuity of functions of one or more variables.

Differentiation, Leibnitz formula, maxima and minima, Taylor's theorem. Differentiation of functions of several variables. Applications of differential Calculus.

Indefinite integral, Fundamental theorem of Calculus, Riemann integration and properties. Improper integrals. Differentiation under the integral sign. Double and multiple integrals and Applications.

Syllabus for Statistics

Probability and Sampling Distributions

Notions of sample space and probability, combinatorial probability, conditional probability and independence, random variable and expectations, moments, standard discrete and continuous distributions, sampling distributions of statistics based on normal samples, central limit theorem, approximation of Binomial to Normal or Poisson law. Bivariate normal and multivariate normal distributions.

Descriptive Statistics

Descriptive statistical measures, graduation of frequency curves, product-moment, partial and multiple correlation, Regression (bivariate and multivariate).

Inference

Elementary theory and methods of estimation (unbiasedness, minimum variance, sufficiency, maximum likelihood method, method of moments). Tests of hypotheses (basic concepts and simple applications of Neyman-Pearson Lemma). Confidence intervals. Tests of regression. Elements of non-parametric inference.

Design of Experiments and Sample Surveys

Basic designs (CRD/RBD/LSD) and their analyses. Elements of factorial designs. Conventional sampling techniques (SRSWR/SRSWOR) including stratification; ratio and regression methods of estimation.

Sample Questions

Group A

1. Suppose two teams play a series of games, each producing a winner and a loser, until one team has won two more games than the other. Let G be the total number of games played. Assume each team has a chance of 0.5 to win each game, independent of the results of the previous games.

   (a) Find the probability distribution of G.

   (b) Find the expected value of G.

2. Is the following system of equations always consistent for real k? Justify your answer.

$$\begin{array}{ccccccc} x & + & y & + & kz & = & 2, \\ 3x & + & 4y & + & 2z & = & k, \\ 2x & + & 3y & - & z & = & 1. \end{array}$$

Find the value of k for which this system admits more than one solution? Express the general solution for the system of equations for this value of k.

Group B

3. Suppose a random vector (X, Y) has joint probability density function

$$f(x, y) = 3y$$

on the triangle bounded by the lines $y = 0$, $y = 1 - x$, and $y = 1 + x$. Compute $E(Y | X \leq \frac{1}{2})$.

4. Two policemen are sent to watch a road that is 1 km long. Each of the two policemen is assigned a position on the road which is chosen according to a uniform distribution along the length of the road and independent of the other's position. Find the probability that the policemen will be less than 1/4 kilometer apart when they reach their assigned posts.

5. Here is a partial key-block of a 2^4 factorial experiment (with factors A, B, C, D) conducted in two blocks of size 8 each:

Partial key-block: $ad \quad bd \quad c \quad \cdots$

Search out the other five treatment combinations for the key-block and also the confounded interaction. Also, give the treatment combination of the second block.

6. Let Y_1, Y_2, Y_3 and Y_4 be four uncorrelated random variables with

$$E(Y_i) = i\theta, \quad \text{Var}\,(Y_i) = i^2\sigma^2, \quad i = 1, 2, 3, 4,$$

where θ and σ (> 0) are unknown parameters. Find the values of c_1, c_2, c_3 and c_4 for which $\sum_{i=1}^4 c_i Y_i$ is unbiased for θ and has least variance.

7. Suppose X has a normal distribution with mean 0 and variance 25. Let Y be an independent random variable taking values -1 and 1 with equal probability. Define $S = XY + \frac{X}{Y}$ and $T = XY - \frac{X}{Y}$.

   (a) Find the probability distribution of S.

   (b) Find the probability distribution of $(\frac{S+T}{10})^2$.

8. Let $X_1, X_2, \ldots X_n$ be i. i. d. with common density $f(x;\theta)$ given by

$$f(x;\theta) = \frac{1}{2\theta}\exp(-|x|/\theta), \quad -\infty < x < \infty, \quad \theta \in (0, \infty).$$

In case of each of the statistics S and T defined below, decide (a) if it is an unbiased estimator of θ, (b) if it is an MLE for θ and (c) if it is sufficient for θ. Give reasons.

$$S = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad T = \frac{1}{n}\sum_{i=1}^{n} |X_i|.$$

9. Let Y_1, Y_2, Y_3 and Y_4 be a random sample from a population with

probability density function

$$f(y,\theta) = \begin{cases} \left(\frac{1}{2\theta^3}\right) y^2 \exp(-y/\theta) & \text{if } y > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Find the most powerful test for testing $H_0 : \theta = \theta_0$ against the hypothesis $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$. Is the test uniformly most powerful for $\theta > \theta_0$?

10. On a particular day let X_1, X_2 and X_3 be the number of boys born before the first girl is born in hospitals 1, 2 and 3 respectively. If the observations are $X_1 = 0$, $X_2 = 3$ and $X_3 = 2$, find the most powerful test to test the null hypothesis that a girl and a boy are equally likely to be born against the alternative that a girl is less likely to be born than a boy.

11. For $n \geq 1$ let $x_n = \dfrac{1}{n\alpha_n}$ where α_n is such that $2^{\alpha_n - 2} < n \leq 2^{\alpha_n - 1}$. Is the series $\sum_{n=1}^{\infty} x_n$ convergent or divergent? Justify your answer.

12. Let f and g be continuous functions such that $f(x) \leq g(x)$ for all $x \in [0,1]$. Determine which of the following statements are true and which are false:

    (a) $\int_0^x |f(t)|dt \leq \int_0^x |g(t)|dt$ for all $x \in [0,1]$,

    (b) $\int_0^x (|f(t)| + f(t))dt \leq \int_0^x (|g(t)| + g(t))dt$ for all $x \in [0,1]$,

    (c) $\int_0^x (|f(t)| - f(t))dt \leq \int_0^x (|g(t)| - g(t))dt$ for all $x \in [0,1]$.

    For any statement which you believe to be true, you need to give a proof and for any statement which you believe to be false, you need to give a counter example.