INSTITUTE OF ACTUARIES OF INDIA

EXAMINATIONS

31st October 2007

Subject CT4– Models

Time allowed: Three Hours (10.00 am – 13.00 Hrs)

Total Marks: 100

INSTRUCTIONS TO THE CANDIDATES

1. ***Do not write your name anywhere on the answer sheet/s. You have only to write your Candidate's Number on each answer sheet/s.***
2. ***Mark allocations are shown in brackets.***
3. ***Attempt all questions, beginning your answer to each question on a separate sheet.***
4. ***Fasten your answer sheets together in numerical order of questions. This, you may complete immediately after expiry of the examination time.***
5. ***In addition to this paper you should have available graph paper, Actuarial Tables and an electronic calculator.***

Professional Conduct:

"It is brought to your notice that in accordance with provisions contained in the Professional Conduct Standards, If any candidate is found copying or involved in any other form of malpractice, during or in connection with the examination, Disciplinary action will be taken against the candidate which may include expulsion or suspension from the membership of ASI."

Candidates are advised that a reasonable standard of handwriting legibility is expected by the examiners and that candidates may be penalized if undue effort is required by the examiners to interpret scripts.

AT THE END OF THE EXAMINATION

Please return your answer sheet/s and this question paper to the supervisor separately

Q. 1) For discrete time stochastic process Xn, define the following terms:

- **i).** Stationary
- **ii).** Weakly stationary
- **iii).** Increment
- **iv).** Markov property

[4]

Q. 2) Stochastic models are classified as discrete or continuous time variable and discrete or continuous state space giving rise to four classifications. Give two examples of each of the four types of stochastic model, which may be used to model the observed process. **[4]**

Q. 3) In a mortality investigation the actual number of deaths at age x last birthday is d_x .The goodness of fit between the data and the force of mortality $\mu_{x+1/2}$ over the age range x_1 , $x_1 + 1$, $x_1 + 2$ ..., $x_1 + m - 1$ can be tested using the statistic

$$\sum_{x=x_1}^{x=x+m-1} (d_x - E_x^c \mu_{x+1/2})^2 / E_x^c \; \mu_{x+1/2}$$

Where E_x^c is the central exposed to risk, which corresponds to d_x

For the each of the following cases state the null hypothesis being tested and the sampling distribution of this statistic.

a). if $\mu_{x+1/2}$ is taken from a standard mortality table

b). if $\mu_{x+1/2}$ are graduated rates obtained from the crude estimates using the gompertz- makeham formula

$$g(x) = a_0 + \exp\{ b_0 + b_1 x + b_2 x^2 \}$$

[4]

Q. 4) A credit ratings company assessed the credit worthiness of various firms every quarter; the ratings (in decreasing order of merit) are A, B, C and D (default). The rating company has analyzed historical data and has come to the conclusion that the credit rating a typical firm evolves a Markov chain with the following transition matrix:

$$P = \begin{pmatrix} 1-\alpha-\alpha^2 & \alpha & \alpha^2 & 0 \\ \alpha & 1-2\alpha-\alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & \alpha & 1-2\alpha-\alpha^2 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for some parameter α.

i). Determine the range of values of "α" for which the matrix P is a valid transition matrix. (2)
ii). State with reasons whether the chain is irreducible and a periodic. (2)
[4]

Q. 5) A study has been conducted to investigate the effect of a newly invented drug on a group of patients who are suffering from cancer. The following proportional hazard regression model has been fitted to the mortality data of the group of patients.

$h_i(t) = h_0(t) \times \exp\{0.01(x_i - 30) + 0.2 y_i - 0.05 z_i\}$ where

$\mathbf{h_i(t)}$ denotes the hazard function for life i at duration t

$\mathbf{h_0(t)}$; denotes the baseline hazard function at duration t

$\mathbf{x_i}$;denotes the age at entry into the observation of life i

$\mathbf{y_i}$ = 1 if life i is a non – smoker, else 0.

$\mathbf{z_i}$ = 1 if life i is a male , 0 if female .

(a) Describe the class of lives to which the baseline hazard function applies. (1)
(b) Using the model compare the survival function of a male smoker aged 30 at entry relative to a female smoker aged 40 at entry. (2)
(c) Using the model compare the survival function of a male smoker aged 30 at entry relative to a male smoker aged 40 at entry. (2)
[5]

Q. 6) Let us consider a homogeneous Markov chain with state space S = {1, 2, 3} and the corresponding transition matrix is given by

$$P = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$

Using this information,
i). Calculate the three-step transition matrix. (2)

ii). Calculate, for each of the following initial conditions, the probability that the chain will be in state 3 when it is observed at time n = 3 given that:
a). The chain is in state 1 at time zero and in state 2 at time 1
b). The probabilities, of being in states 1, 2 and 3 at time zero, are given by 14/31, 9/31 and 8/31 respectively. (3)

iii). Comment on how the answers to (a) and (b) in part (ii.) would change if the time of the observation was n = 300 instead of n = 3? (3)

[8]

Q. 7)

i) In a mortality investigation the observed number of deaths at age *x last* birthday is d_x and the corresponding initial exposed to risk is E_x. If q_x is the true initial rate of mortality at age x and θ_x is the random variable representing the number of deaths at age **x** last birthday, show that the standard error of the estimator

$$\tilde{q}_x = \theta_x / E_x$$

is estimated approximately by

$$\sqrt{d_x} / E_x$$

stating clearly the assumptions you make in deriving this result. (3)

ii) Describe the implications of this result for the estimation of the mortality rates for assured lives over the age range 18 to 110. (2)

iii) Explain how the interval

$$\overset{0}{q}_x \pm 2\sqrt{d_x} / E_x$$

might be used to assist in the graphical graduation of a set of observed mortality rates $\overset{0}{q}_x$, where

$$\overset{0}{q}_x = d_x / E_x$$

(3)

[8]

Q. 8) Suppose that the future lifetime of a life aged x (x > 0) is represented by a random variable T_x distributed on the interval (0, w-x), where w is some limiting age.

(i) For $0 \le x < y < w$, state a consistency condition between T_x and T_y. (1)

(ii) Suppose that $0 \le x < y < w$, and let $t = y - x$. Define the force of mortality at age y:

a) in terms of T_0; and

b) in terms of T_x

c). Show that the two definitions are equivalent. (6)

(iii) Prove that ${}_tp_x = exp\{-\int_0^t \mu_{x+s} ds\}$ (2)

[9]

Q. 9) A large life office has decided to investigate its non-select mortality experience over the period 1 January 1997 to 31 December 2006. The number of in force policyholders classified by age next birthday at entry plus nearest duration at the investigation date is available at each 1 January from 1997 to 2007. The number of deaths classified by age next birthday at entry plus curtate duration at the date of death is available for each inter-investigation period.

i) State, with reasons, the rate interval you would use for this investigation. State the age definition you would use, and indicate the exact age to which your estimated mortality rates would apply. (2)

ii) Derive an expression for the initial exposed to risk corresponding to your answer to (i). All other assumptions should be stated. (4)

iii) State with reasons how your answers to (i) and (ii) would be called into question by each of the following additional pieces of information:

a) Policies were affected on average three months before a birthday.

b) Policies were affected on average at the end of February in each calendar year. (5)

[11]

Q. 10) **i)** Describe the following graduation tests, specifically the relevant formulae, and commenting briefly on their role in the graduation process.

(a) Smoothness test. (2)

(b) Chi-squared test. (3)

(c) Serial correlations test (with lag 1). (4)

ii) Describe briefly how you would perform a graduation of crude mortality data using a graphical method. Explain how you would attempt to achieve the required criteria of smoothness and adherence to data using this method. Details of graduation tests are not required. (3)

iii) Give examples of the kinds of circumstances under which the Graphic method would be used in preference to graduation by reference to a standard table, or by mathematical formula, giving brief reasons. (1)

[13]

Q. 11) An automobile insurance company XYZ Limited operates a no claims discount system with the following four levels:

Level 1 : 0% discount
Level 2 : 25% discount
Level 3 : 40% discount
Level 4 : 50% discount

The rules for moving between these levels are as follows:

a) A policyholder enters at Level 1.
b) Following a claim free year, move to the next higher level, or remain at Level 4
c) Following a year with one or more claims

- Move back one level or stay at Level 1, if, in the year before the most recent year, there were no claims,
- Move back two levels or move to Level 1 or stay at Level 1, if, in the year before the most recent year, there was one or more claims.

It has also been observed that for a given policyholder, the probability of no claims in a year is 0.8.
Now:

i). Let X(t) denote the discount level, either 1, 2, 3 or 4 of the policyholder in year t. Is $\{X(t)\}_{t=1}^{\infty}$ a Markov chain? Explain your answer. (2)

ii). Also,

a). By increasing the number of levels, define a new stochastic process $\{Y(t)\}_{t=1}^{\infty}$ which is Markov and is such that Y(t) indicates the discount level for the policyholder in year t. (4)

b). Write down the transition matrix for the Markov chain $\{Y(t)\}_{t=1}^{\infty}$. (4)

c). Calculate the long run probability of a policyholder being in discount Level 3. (4)

[14]

Q. 12) A car factory uses "m" assembling machines at a time to assemble a car. Each machine has a probability of "$\mu dt + o(dt)$" of breaking down in any time interval of "dt". A mechanic is constantly repairing the impaired machines, so that in any time interval "dt", there is a probability of "$\lambda dt + o(dt)$" that he will send back one of the machines to the assembly line after its repair, irrespective of the number of machines he has yet to repair.

i). Write down the generator matrix. (6)

ii). Show that the stationary distribution $\{\pi_n\}$ satisfies

$$n \mu \pi_n - \lambda \pi_{n-1} = (n+1) \mu \pi_{n+1} - \lambda \pi_n, \qquad n = 1, 2, \ldots, m-1$$ (5)

iii). Hence or otherwise show that

$$\pi_n = k \frac{(\lambda / \mu)^n}{n!}$$

Where "k" is a constant that you have to determine. (5)

[16]

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