Test Codes: SIA (Multiple-choice Type) and SIB (Short Answer Type) 2009

Questions will be set on the following and related topics.

Algebra: Sets, operations on sets. Prime numbers, factorization of integers and divisibility. Rational and irrational numbers. Permutations and combinations. Binomial theorem. Logarithms. Theory of quadratic equations. Polynomials and remainder theorem. Arithmetic, geometric and harmonic progressions. Complex numbers. Algebraic inequalities.

Geometry: Plane geometry of class X level. Geometry of 2 dimensions with Cartesian and polar coordinates. Concept of a locus. Equation of a line, angle between two lines, distance from a point to a line. Area of a triangle. Equations of circle, parabola, ellipse and hyperbola and equations of their tangents and normals. Mensuration.

Trigonometry: Measures of angles. Trigonometric and inverse trigonometric functions. Trigonometric identities including addition formulæ, solutions of trigonometric equations. Properties of triangles. Heights and distances.

Calculus: Functions, one-one functions, onto functions. Limits and continuity. Derivatives and methods of differentiation. Slope of a curve. Tangents and normals. Maxima and minima. Curve-tracing using calculus. Methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Logical Reasoning.

Note. The actual selection paper will have 30 questions in SIA and 10 questions in SIB. The questions in SIA will be multiple choice-type. The questions in SIB will be short answer-type. The questions in SIA will be divided into two groups, A and B. All questions in both the groups will have 4 options. However, Group A will consist of 20 questions and *exactly one* option will be correct. Group B will consist of 10 questions and each question will have *either one or two* correct options. In either case, the student will have to identify *all* the correct options and *only* the correct options in order to get credit for that question.

Sample Questions for SIA

Group A

Each of the following questions have exactly one correct option and you have to identify it.

1. The last digit of $(2137)^{754}$ is

   (A) 1. (B) 3. (C) 7. (D) 9.

2. The sum of all distinct four digit numbers that can be formed using the digits $1, 2, 3, 4$, and 5, each digit appearing at most once, is

   (A) 399900. (B) 399960. (C) 390000. (D) 360000.

3. The coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$ is

   (A) $\frac{12!}{3!4!5!}$. (B) $\binom{6}{3}3!$. (C) 33. (D) $3\binom{6}{3}$.

4. If $\log_{10} x = 10^{\log_{100} 4}$ then x equals

   (A) 4^{10}. (B) 100. (C) $\log_{10} 4$. (D) none of the above.

5. Let $ABCD$ be a unit square. Four points E, F, G and H are chosen on the sides AB, BC, CD and DA respectively. The lengths of the sides of the quadrilateral $EFGH$ are α, β, γ and δ. Which of the following is always true?

   (A) $1 \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 2\sqrt{2}$.
   (B) $2\sqrt{2} \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 4\sqrt{2}$.
   (C) $2 \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 4$.
   (D) $\sqrt{2} \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 2 + \sqrt{2}$.

6. The set of all real numbers x satisfying the inequality $x^3(x+1)(x-2) > 0$ is

   (A) the interval $(2, \infty)$. (B) the interval $(0, \infty)$.
   (C) the interval $(-1, \infty)$. (D) none of the above.

7. The sides of a triangle are given to be $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Then the largest of the three angles of the triangle is

   (A) $75°$. (B) $\left(\frac{x}{x+1}\pi\right)$ radians. (C) $120°$. (D) $135°$.

8. Two poles, AB of length two metres and CD of length twenty metres are erected vertically with bases at B and D. The two poles are at a distance not less than twenty metres. It is observed that $\tan \angle ACB = 2/77$. The distance between the two poles is

   (A) $72m$. (B) $68m$. (C) $24m$. (D) $24.27m$.

9. z_1, z_2 are two complex numbers with $z_2 \neq 0$ and $z_1 \neq z_2$ and satisfying $\left|\dfrac{z_1 + z_2}{z_1 - z_2}\right| = 1$. Then $\dfrac{z_1}{z_2}$ is

   (A) real and negative.
   (B) real and positive.
   (C) purely imaginary.
   (D) none of the above need to be true always.

10. Let A be the fixed point $(0, 4)$ and B be a moving point $(2t, 0)$. Let M be the mid-point of AB and let the perpendicular bisector of AB meet the y-axis at R. The locus of the mid-point P of MR is

    (A) $y + x^2 = 2$. (B) $x^2 + (y - 2)^2 = 1/4$.
    (C) $(y - 2)^2 - x^2 = 1/4$. (D) none of the above.

11. If A, B, C are the angles of a triangle and $\sin^2 A + \sin^2 B = \sin^2 C$, then C is equal to

    (A) $30°$. (B) $90°$. (C) $45°$. (D) none of the above.

12. In the interval $(-2\pi, 0)$, the function $f(x) = \sin\left(\dfrac{1}{x^3}\right)$

    (A) never changes sign.
    (B) changes sign only once.
    (C) changes sign more than once, but finitely many times.
    (D) changes sign infinitely many times.

13. The limit

$$\lim_{x \to 0} \frac{(e^x - 1)\tan^2 x}{x^3}$$

    (A) does not exist. (B) exists and equals 0.
    (C) exists and equals 2/3. (D) exists and equals 1.

14. Let $f_1(x) = e^x$, $f_2(x) = e^{f_1(x)}$ and generally $f_{n+1}(x) = e^{f_n(x)}$ for all $n \geq 1$. For any fixed n, the value of $\dfrac{d}{dx} f_n(x)$ is equal to

(A) $f_n(x)$. (B) $f_n(x)f_{n-1}(x)$.
(C) $f_n(x)f_{n-1}(x)\cdots f_1(x)$. (D) $f_{n+1}(x)f_n(x)\cdots f_1(x)e^x$.

15. If the function

$$f(x) = \begin{cases} \frac{x^2-2x+A}{\sin x} & \text{if } x \neq 0 \\ B & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$, then

(A) $A = 0,\ B = 0$. (B) $A = 0,\ B = -2$.
(C) $A = 1,\ B = 1$. (D) $A = 1,\ B = 0$.

16. A truck is to be driven 300 kilometres (kms.) on a highway at a constant speed of x kms. per hour. Speed rules of the highway require that $30 \leq x \leq 60$. The fuel costs ten rupees per litre and is consumed at the rate $2 + (x^2/600)$ litres per hour. The wages of the driver are 200 rupees per hour. The most economical speed (in kms. per hour) to drive the truck is

(A) 30. (B) 60. (C) $30\sqrt{3.3}$. (D) $20\sqrt{33}$.

17. If $b = \displaystyle\int_0^1 \frac{e^t}{t+1}dt$ then $\displaystyle\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}dt$ is

(A) be^a. (B) be^{-a}. (C) $-be^{-a}$. (D) $-be^a$.

18. In the triangle ABC, the angle $\angle BAC$ is a root of the equation

$$\sqrt{3}\cos x + \sin x = 1/2.$$

Then the triangle ABC is

(A) obtuse angled. (B) right angled.
(C) acute angled but not equilateral. (D) equilateral.

19. The number of maps f from the set $\{1, 2, 3\}$ into the set $\{1, 2, 3, 4, 5\}$ such that $f(i) \leq f(j)$ whenever $i < j$ is

(A) 60. (B) 50. (C) 35. (D) 30.

20. Let n be a positive integer. Consider a square S of side $2n$ units. Divide S into $4n^2$ unit squares by drawing $2n-1$ horizontal and $2n-1$ vertical lines one unit apart. A circle of diameter $2n-1$ is drawn with its centre at the intersection of the two diagonals of the square S. How many of these unit squares contain a portion of the circumference of the circle?

(A) $4n - 2$. (B) $4n$. (C) $8n - 4$. (D) $8n - 2$.

21. A lantern is placed on the ground 100 feet away from a wall. A man six feet tall is walking at a speed of 10 feet/second from the lantern to the nearest point on the wall. When he is midway between the lantern and the wall, the rate of change (in ft./sec.) in the length of his shadow is

(A) 2.4. (B) 3. (C) 3.6. (D) 12.

22. An isosceles triangle with base 6 cms. and base angles 30° each is inscribed in a circle. A second circle touches the first circle and also touches the base of the triangle at its midpoint. If the second circle is situated outside the triangle, then its radius (in cms.) is

(A) $3\sqrt{3}/2$. (B) $\sqrt{3}/2$. (C) $\sqrt{3}$. (D) $4/\sqrt{3}$.

23. Let n be a positive integer. Define

$$f(x) = \min\{|x-1|, |x-2|, \ldots, |x-n|\}.$$

Then $\int_0^{n+1} f(x)dx$ equals

(A) $\frac{(n+4)}{4}$. (B) $\frac{(n+3)}{4}$. (C) $\frac{(n+2)}{2}$. (D) $\frac{(n+2)}{4}$.

24. Let $S = \{1, 2, \ldots, n\}$. The number of possible pairs of the form (A, B) with $A \subseteq B$ for subsets A and B of S is

(A) 2^n. (B) 3^n. (C) $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$. (D) $n!$.

25. Consider three boxes, each containing 10 balls labelled 1, 2, ..., 10. Suppose one ball is drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the i-th box, $i = 1, 2, 3$. Then the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$ is

(A) 120. (B) 130. (C) 150. (D) 160.

26. The maximum of the areas of the isosceles triangles with base on the positive x-axis and which lie below the curve $y = e^{-x}$ is:

(A) $1/e$. (B) 1. (C) 1/2. (D) e.

27. Suppose a, b and n are positive integers, all greater than one. If a^n+b^n is prime, what can you say about n?

(A) The integer n must be 2.
(B) The integer n need not be 2, but must be a power of 2.
(C) The integer n need not be a power of 2, but must be even.
(D) None of the above is necessarily true.

28. Water falls from a tap of circular cross section at the rate of 2 metres/sec and fills up a hemispherical bowl of inner diameter 0.9 metres. If the inner diameter of the tap is 0.01 metres, then the time needed to fill the bowl is

(A) 40.5 minutes. (B) 81 minutes.
(C) 60.75 minutes. (D) 20.25 minutes.

29. The value of the integral

$$\int_{\pi/2}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}}\, dx$$

equals

(A) 1. (B) π. (C) e. (D) none of these.

30. The set of all solutions of the equation $\cos 2\theta = \sin\theta + \cos\theta$ is given by

(A) $\theta = 0$.
(B) $\theta = n\pi + \frac{\pi}{2}$, where n is any integer.
(C) $\theta = 2n\pi$ or $\theta = 2n\pi - \frac{\pi}{2}$ or $\theta = n\pi - \frac{\pi}{4}$, where n is any integer.
(D) $\theta = 2n\pi$ or $\theta = n\pi + \frac{\pi}{4}$, where n is any integer.

31. For $k \geq 1$, the value of

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k}$$

equals

(A) $\binom{n+k+1}{n+k}$. (B) $(n+k+1)\binom{n+k}{n+1}$.
(C) $\binom{n+k+1}{n+1}$. (D) $\binom{n+k+1}{n}$.

32. The value of

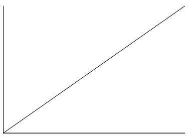
$$\sin^{-1}\cot\left[\sin^{-1}\left\{\frac{1}{2}\left(1-\sqrt{\frac{5}{6}}\right)\right\} + \cos^{-1}\sqrt{\frac{2}{3}} + \sec^{-1}\sqrt{\frac{8}{3}}\right]$$

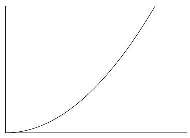
is

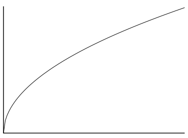
(A) 0. (B) $\pi/6$. (C) $\pi/4$. (D) $\pi/2$.

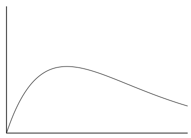
33. Which of the following graphs represents the function

$$f(x) = \int_0^{\sqrt{x}} e^{-u^2/x} du, \quad \text{for} \quad x > 0 \quad \text{and} \quad f(0) = 0?$$

(A) 

(B) 

(C) 

(D) 

34. If $a_n = \left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)^2\left(1 + \frac{3^2}{n^2}\right)^3 \cdots \left(1 + \frac{n^2}{n^2}\right)^n$, then

$$\lim_{n\to\infty} a_n^{-1/n^2}$$

is

(A) 0. (B) 1. (C) e. (D) $\sqrt{e}/2$.

35. If $f(x) = e^x \sin x$, then $\frac{d^{10}}{dx^{10}} f(x) \Big|_{x=0}$ equals

(A) 1. (B) -1. (C) 10. (D) 32.

36. Consider a circle with centre O. Two chords AB and CD extended intersect at a point P outside the circle. If $\angle AOC = 43°$ and $\angle BPD = 18°$, then the value of $\angle BOD$ is

(A) 36°. (B) 29°. (C) 7°. (D) 25°.

37. A box contains 10 red cards numbered $1, \ldots, 10$ and 10 black cards numbered $1, \ldots, 10$. In how many ways can we choose 10 out of the 20 cards so that there are exactly 3 *matches*, where a *match* means a red card and a black card with the same number?

(A) $\binom{10}{3}\binom{7}{4}2^4$. (B) $\binom{10}{3}\binom{7}{4}$.

(C) $\binom{10}{3}2^7$. (D) $\binom{10}{3}\binom{14}{4}$.

38. Let P be a point on the ellipse $x^2 + 4y^2 = 4$ which does not lie on the axes. If the normal at the point P intersects the major and minor axes at C and D respectively, then the ratio $PC : PD$ equals

(A) 2. (B) 1/2. (C) 4. (D) 1/4.

39. The set of complex numbers z satisfying the equation

$$(3 + 7i)z + (10 - 2i)\bar{z} + 100 = 0$$

represents, in the Argand plane,

(A) a straight line.
(B) a pair of intersecting straight lines.
(C) a pair of distinct parallel straight lines.
(D) a point.

40. The number of triplets (a, b, c) of integers such that $a < b < c$ and a, b, c are sides of a triangle with perimeter 21 is

(A) 7. (B) 8. (C) 11. (D) 12.

41. Suppose a, b and c are three numbers in G.P. If the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then $\frac{d}{a}$, $\frac{e}{b}$ and $\frac{f}{c}$ are in

(A) A.P. (B) G.P. (C) H.P. (D) none of the above.

42. The number of solutions of the equation $\sin^{-1} x = 2\tan^{-1} x$ is

(A) 1. (B) 2. (C) 3. (D) 5.

43. Suppose $ABCD$ is a quadrilateral such that $\angle BAC = 50°$, $\angle CAD = 60°$, $\angle CBD = 30°$ and $\angle BDC = 25°$. If E is the point of intersection of AC and BD, then the value of $\angle AEB$ is

(A) 75°. (B) 85°. (C) 95°. (D) 110°.

44. Let $\mathbb{R}$ be the set of all real numbers. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x^2 + 6x - 5$ is

(A) one-to-one, but not onto. (B) one-to-one and onto.
(C) onto, but not one-to-one. (D) neither one-to-one nor onto.

45. Let L be the point $(t, 2)$ and M be a point on the y-axis such that LM has slope $-t$. Then the locus of the midpoint of LM, as t varies over all real values, is

(A) $y = 2 + 2x^2$. (B) $y = 1 + x^2$.
(C) $y = 2 - 2x^2$. (D) $y = 1 - x^2$.

46. Suppose $x, y \in (0, \pi/2)$ and $x \neq y$. Which of the following statements is true?

(A) $2\sin(x + y) < \sin 2x + \sin 2y$ for all x, y.
(B) $2\sin(x + y) > \sin 2x + \sin 2y$ for all x, y.
(C) There exist x, y such that $2\sin(x + y) = \sin 2x + \sin 2y$.
(D) None of the above.

47. A triangle ABC has a fixed base BC. If $AB : AC = 1 : 2$, then the locus of the vertex A is

(A) a circle whose centre is the midpoint of BC.
(B) a circle whose centre is on the line BC but not the midpoint of BC.
(C) a straight line.
(D) none of the above.

48. Let N be a 50 digit number. All the digits except the 26th one from the right are 1. If N is divisible by 13, then the unknown digit is

(A) 1. (B) 3. (C) 7. (D) 9.

49. Suppose $a < b$. The maximum value of the integral

$$\int_a^b \left(\frac{3}{4} - x - x^2\right) dx$$

over all possible values of a and b is

(A) 3/4. (B) 4/3. (C) 3/2. (D) 2/3.

50. For any $n \geq 5$, the value of $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1}$ lies between

(A) 0 and $n/2$. (B) $n/2$ and n.
(C) n and $2n$. (D) none of the above.

51. Let ω denote a cube root of unity which is not equal to 1. Then the number of distinct elements in the set

$$\{(1+\omega+\omega^2+\cdots+\omega^n)^m \quad : \quad m,n=1,2,3,\cdots\}$$

is

(A) 4. (B) 5. (C) 7. (D) infinite.

52. The value of the integral

$$\int_2^3 \frac{dx}{\log_e x}$$

(A) is less than 2.
(B) is equal to 2.
(C) lies in the interval $(2,3)$.
(D) is greater than 3.

53. For each positive integer n, define a function f_n on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} 0, & \text{if } x=0, \\ \sin\frac{\pi}{2n}, & \text{if } 0 < x \le \frac{1}{n}, \\ \sin\frac{2\pi}{2n}, & \text{if } \frac{1}{n} < x \le \frac{2}{n}, \\ \sin\frac{3\pi}{2n}, & \text{if } \frac{2}{n} < x \le \frac{3}{n}, \\ \vdots & \\ \sin\frac{n\pi}{2n}, & \text{if } \frac{n-1}{n} < x \le 1. \end{cases}$$

Then, the value of $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx$ is

(A) π. (B) 1. (C) $\frac{1}{\pi}$. (D) $\frac{2}{\pi}$.

54. In a win-or-lose game, the winner gets 2 points whereas the loser gets 0. *Six* players A, B, C, D, E and F play each other in a preliminary round from which the top *three* players move to the final round. After each player has played *four* games, A has 6 points, B has 8 points and C has 4 points. It is also known that E won against F. In the next set of games D, E and F win their games against A, B and C respectively. If A, B and D move to the final round, the final scores of E and F are, respectively,

(A) 4 and 2. (B) 2 and 4. (C) 2 and 2. (D) 4 and 4.

55. The number of ways in which one can select six distinct integers from the set $\{1, 2, 3, \cdots, 49\}$, such that no two consecutive integers are selected, is

(A) $\binom{49}{6} - 5\binom{48}{5}$. (B) $\binom{43}{6}$. (C) $\binom{25}{6}$. (D) $\binom{44}{6}$.

Group B

Each of the following questions has either one or two correct options and you have to identify all the correct options.

1. If positive numbers a, b, c, d are such that $1/a, 1/b, 1/c, 1/d$ are in arithmetic progression then we always have

   (A) $a + d \geq b + c$. (B) $a + b \geq c + d$.
   (C) $a + c \geq b + d$. (D) $bc > ad$.

2. Let $n > 3$ be odd. Then

   (A) $n^2 - 1$ is always divisible by 16.
   (B) If $n - 1$ is a power of 2, then $n^2 - 1$ can be written as mq, where m is a power of 2 and q is an odd integer.
   (C) $n^2 - 1$ is always divisible by 24, when n is a prime.
   (D) There is no value of n such that $2^{57912} + 1$ divides $n^2 - 1$.

3. ABC is a right angled triangle with hypotenuse BC. Then

   (A) the vertex A, the circumcentre, the centroid, and the orthocentre of the triangle always lie on a straight line.
   (B) the vertex A, the orthocentre, the circumcentre, and the incentre of the triangle always lie on a straight line.
   (C) the vertex A, the circumcentre, and the centroid and the incentre of the triangle always lie on a straight line.
   (D) the circumcentre, the centroid, the incentre and the orthocentre of the triangle always lie on a straight line.

4. Consider the function $f(x) = e^{2x} - x^2$. Then

   (A) $f(x) = 0$ for some $x < 0$. (B) $f(x) = 0$ for some $x > 0$.
   (C) $f(x) \neq 0$ for every $x < 0$. (D) $f(x) \neq 0$ for every $x > 0$.

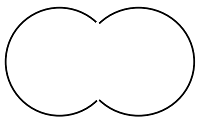
5. Let Q be a quadrilateral such that two of its sides are of length a and two other sides are of length b. If the area of Q is maximized, then

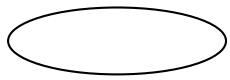
(A) area of $Q = ab$. (B) area of $Q > ab$.
(C) Q must be a parallelogram. (D) Q must be a rectangle.

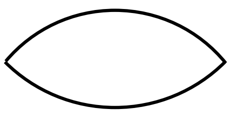
6. The expression $x^5 + x + 1$ has
   (A) three distinct real roots.
   (B) only one real root which has multiplicity 3.
   (C) only one real root which has multiplicity 1.
   (D) no integer root.

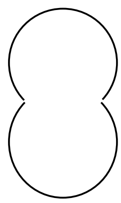
7. Four passengers in a compartment of the Delhi-Howrah Rajdhani Express discover that they form an interesting group. Two are lawyers and two are doctors. Two of them speak Bengali and the other two Hindi and no two of the same profession speak the same language. Two are also Christians and two are Muslims. No two of the same religion are of the same profession, neither do they speak the same language. Hindi-speaking doctor is a Christian.
   (A) Bengali-speaking lawyer is a Muslim.
   (B) Christian lawyer speaks Bengali.
   (C) Bengali-speaking doctor is a Christian.
   (D) Hindi-speaking lawyer is a Muslim.

8. Let AB be a fixed line segment. Let P be a moving point such that $\angle APB$ is equal to a constant **ACUTE** angle. Then, which of the following curves does the point P move along?

(A) 

(B) 

(C) 

(D) 

9. Every integer of the form $(n^3 - n)(n - 2)$ for $n = 3, 4, \dots$ is
   (A) always divisible by 12. (B) always divisible by 24.
   (C) always divisible by 48. (D) always divisible by 18.

10. Let $f(x) = 2 + \cos x$ for all real x.
   Statement 1: For each real t, there exists a point c in $[t, t + \pi]$ such that $f'(c) = 0$.

Statement 2: For each real t, $f(t) = f(t + 2\pi)$ holds.

(A) Statement 1 is TRUE and Statement 2 is also the CORRECT REASON for Statement 1.

(B) Statement 1 is TRUE and Statement 2 is NOT the CORRECT REASON for Statement 1.

(C) Statement 1 is TRUE and Statement 2 is FALSE.

(D) Statement 1 is FALSE and Statement 2 is TRUE.

11. Let $a_1 = b_1 = 1$, $a_2 = 2$, $b_2 = 3$, $a_{n+1} = a_n + b_n$, $b_{n+1} = 2a_n + b_n$. Then

(A) $b_n^2 = 2a_n^2 + 1$, if n is odd. (B) $b_n^2 = 2a_n^2 - 1$, if n is odd.

(C) $\lim_{n\to\infty} \frac{b_n}{a_n} = \sqrt{2} - 1$. (D) $\lim_{n\to\infty} \frac{b_n}{a_n} = \sqrt{2}$.

12. Let $f(x)$ be any differentiable function such that $f(x) = \int_0^x g(y)dy$ and $F(x) = \int_0^x f(y)dy$. Then $\int_0^b x^2 f(x)dx$ equals

(A) $\int_0^b 2x[f(b) - f(x)]dx$. (B) $\int_0^b 2x[F(b) - F(x)]dx$.

(C) $\int_0^b \frac{b^3 - x^3}{3} g(x)dx$. (D) $\int_0^b \left(\frac{b^3}{3} - x^2\right) g(x)dx$.

13. Let f be the function $f(x) = \cos x - 1 + \dfrac{x^2}{2}$. Then

(A) $f(x_0) = 0$ for some $x_0 > 0$.

(B) $f(x_0) = 0$ for some $x_0 < 0$.

(C) $f(x)$ is an increasing function on the interval $(-\infty, 0]$ and decreasing on the interval $[0, \infty)$.

(D) $f(x)$ is a decreasing function on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

14. Let P, Q, R and S be four statements such that if P is true then Q is true, if Q is true then R is true and if S is true then at least one of Q and R is false. It then follows that

(A) if S is false, then both Q and R are true.

(B) if at least one of Q and R is true, then S is false.

(C) if P is true, then S is false.

(D) if S is true, then both P and Q are false.

15. If $\cos B \cos C + \sin B \sin C \sin^2 A = 1$, then $\triangle ABC$ is

(A) isosceles, but not equilateral. (B) right angled.

(C) equilateral. (D) scalene.

16. Let

$$\alpha = \lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3},$$

$$\beta = \lim_{n\to\infty} \frac{(1^3 - 1^2) + (2^3 - 2^2) + \cdots + (n^3 - n^2)}{n^4}.$$

Then

(A) $\alpha = \beta$. (B) $\alpha < \beta$. (C) $4\alpha = 3\beta$. (D) $3\alpha = 4\beta$.

17. Let $x_1, x_2, \ldots, x_{100}$ be hundred integers such that the sum of any five of them is 20. Then

(A) the largest x_i equals 7. (B) the smallest x_i equals 3.
(C) $x_{17} = x_{83}$. (D) the average of all the numbers is 4.

18. Let x be a positive real number. Then

(A) $x^2 + \pi^2 + x^{2\pi} > x\pi + (x + \pi)x^{\pi}$.
(B) $x^{\pi} + \pi^{x} > x^{2\pi} + \pi^{2x}$.
(C) $x\pi + (x + \pi)x^{\pi} > x^2 + \pi^2 + x^{2\pi}$.
(D) $x^3 + \pi^3 + x^{3\pi} > 3\pi x^{\pi+1}$.

19. If a, b and c are three sides of a triangle with perimeter 1, then which of the following is/are always true?

(A) $(1-a)(1-b)(1-c) \le 8abc$. (B) $(1-a)(1-b)(1-c) \le 8/27$.
(C) $bc + ca + ab \le 1/3$. (D) $bc + ca + ab \ge 1/3$.

20. Let x be an irrational number. If a, b, c and d are rational numbers such that $\dfrac{ax+b}{cx+d}$ is a rational number, which of the following must be true?

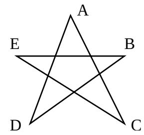
(A) $ad = bc$. (B) $ac = bd$. (C) $ab = cd$. (D) $a = d = 0$.

21. Let $z = x + iy$ be a complex number, which satisfies the equation $(z + \overline{z})z = 2 + 4i$. Then

(A) $y = \pm 2$. (B) $x = \pm 2$. (C) $x = \pm 3$. (D) $y = \pm 1$.

22. In the given figure

(A) $\sin(A + B + C) = \cos(D + E)$.
(B) $\sin(A + B + C) = \sin(D + E)$.
(C) $\cos(A + B + C) = -\sin(D + E)$.
(D) $\cos(A + B + C) = -\cos(D + E)$.



23. Consider the sequence of positive integers 55, 111, 5555, 10101, 555555, 1001001, ... In this sequence

(A) No even term in the sequence is a perfect square.
(B) No odd term in the sequence is a perfect square.
(C) Some even terms are perfect squares.
(D) Some odd terms are perfect squares.

24. Consider the function $f(\theta) = \sin^n \theta + \cos^n \theta$, where $n \geq 3$ is a positive integer. Then

(A) $f(\theta) = 1$ has no real solution in $(0, \pi/2)$.
(B) $f(\theta) = 1$ has exactly one real solution in $(0, \pi/2)$.
(C) $f(\theta) = 1$ has exactly two real solution in $(0, \pi/2)$.
(D) $f(\theta) = 1$ has exactly two real solution in $[0, \pi/2]$.

25. Define $S_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}$, where n is a positive integer. Then

(A) $S_n < \frac{1}{\sqrt{4n+2}}$ for some $n > 2$.
(B) $S_n < \frac{1}{\sqrt{2n+1}}$ for all $n \geq 2$.
(C) $S_n < \frac{1}{\sqrt{2n+5}}$ for all $n \geq 2$.
(D) $S_n > \frac{1}{\sqrt{4n+2}}$ for all $n \geq 2$.

26. The algebraic sum of the perpendicular distances from A (x_1, y_1), B (x_2, y_2), C (x_3, y_3), to a line is zero. Then the line must pass through the

(A) orthocentre of $\triangle ABC$.
(B) centroid of $\triangle ABC$.
(C) incentre of $\triangle ABC$.
(D) circumcentre of $\triangle ABC$.

27. Which of the following is/are contained in the solution set of the equation $4x^2 - 40[x] + 51 = 0$, where x is a real number and $[x]$ denotes the greatest integer that is less than or equal to x?

(A) $13/2 < x < 17/2$.
(B) $3/2 \leq x \leq 7/2$.
(C) $x \neq \sqrt{29}/2$.
(D) None of the above.

28. Let abc be a three digit number in base 10 with $a > c + 1$. Let $abc - cba = efg$. Then

(A) $g = 8$.
(B) $efg + gfe = 1089$.
(C) $f = 9$.
(D) $e = 1$.

29. If $n^2 + 19n + 92$ is a perfect square, then the possible values of n may be

(A) -19.
(B) -8.
(C) -4.
(D) -11.

30. Let a, b and c be three real numbers. Then the equation $\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = 0$

(A) always have real roots.

(B) can have real or complex roots depending on the values of a, b and c.

(C) always have real and equal roots.

(D) always have real roots, which are not necessarily equal.

31. Let X be the set $\{1, 2, 3, \ldots, 10\}$ and P the subset $\{1, 2, 3, 4, 5\}$. The number of subsets Q of X such that

(A) $P \cap Q = \{3\}$ is 1.

(B) $P \cap Q = \{3\}$ is 2^4.

(C) $P \cap Q = \{3\}$ is 2^5.

(D) $P \cap Q = \{3\}$ is 2^9.

32. Suppose that the equations $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have exactly one common non-zero root. Then

(A) $a + b + c = 0$.

(B) the two roots which are not common must necessarily be real.

(C) the two roots which are not common may not be real.

(D) the two roots which are not common are either both real or both not real.

33. Let $\frac{\tan(\alpha-\beta+\gamma)}{\tan(\alpha+\beta-\gamma)} = \frac{\tan\beta}{\tan\gamma}$. Then

(A) $\sin(\beta - \gamma) = \sin(\alpha - \beta)$.

(B) $\sin(\alpha - \gamma) = \sin(\beta - \gamma)$.

(C) $\sin(\beta - \gamma) = 0$.

(D) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

34. Let K be the set of all points (x, y) such that $|x| + |y| \leq 1$. Given a point A in the plane, let F_A be the point in K which is closest to A. Then the points A for which $F_A = (1, 0)$ are

(A) all points $A = (x, y)$ with $x \geq 1$.

(B) all points $A = (x, y)$ with $x \geq y + 1$ and $x \geq 1 - y$.

(C) all points $A = (x, y)$ with $x \geq 1$ and $y = 0$.

(D) all points $A = (x, y)$ with $x \geq 0$ and $y = 0$.

35. Let $\frac{\tan 3\theta}{\tan\theta} = k$. Then

(A) $k \in (1/3, 3)$.

(B) $k \notin (1/3, 3)$.

(C) $\frac{\sin 3\theta}{\sin\theta} = \frac{2k}{k-1}$.

(D) $\frac{\sin 3\theta}{\sin\theta} > \frac{2k}{k-1}$.

Sample Questions for SIB

1. How many natural numbers less than 10^8 are there, with sum of digits equal to 7?

2. Consider the function
$$f(x) = \lim_{n\to\infty} \frac{\log_e(2+x) - x^{2n}\sin x}{1+x^{2n}}$$
defined for $x > 0$. Is $f(x)$ continuous at $x = 1$? Justify your answer. Show that $f(x)$ does not vanish anywhere in the interval $0 \le x \le \frac{\pi}{2}$. Indicate the points where $f(x)$ changes sign.

3. Suppose a is a complex number such that
$$a^2 + a + \frac{1}{a} + \frac{1}{a^2} + 1 = 0.$$
If m is a positive integer, find the value of
$$a^{2m} + a^m + \frac{1}{a^m} + \frac{1}{a^{2m}}.$$

4. Let $f(u)$ be a continuous function and, for any real number u, let $[u]$ denote the greatest integer less than or equal to u. Show that for any $x > 1$,
$$\int_1^x [u]([u]+1)f(u)du = 2\sum_{i=1}^{[x]} i \int_i^x f(u)du.$$

5. Two intersecting circles are said to be orthogonal to each other if the tangents to the two circles at any point of intersection are perpendicular to each other. Show that every circle passing through the points $(2,0)$ and $(-2,0)$ is orthogonal to the circle $x^2 + y^2 - 5x + 4 = 0$.

6. Show that there is exactly one value of x which satisfies the equation
$$2\cos^2(x^3 + x) = 2^x + 2^{-x}.$$

7. An oil-pipe has to connect the oil-well O and the factory F, between which there is a river whose banks are parallel, and not perpendicular to the line joining O and F. The pipe must cross the river perpendicular to the banks. Find the position and nature of the shortest such pipe and justify your answer.

8. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ where $x_1, \cdots, x_n, y_1, \cdots, y_n$ are real numbers. We write $x > y$ if either $x_1 > y_1$ or for some k, with $1 \le k \le n-1$, we have $x_1 = y_1, \ldots, x_k = y_k$, but $x_{k+1} > y_{k+1}$. Show that for $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$, if $u > v$ and $w > z$, then $u + w > v + z$.

9. For any positive integer n, let $f(n)$ be the remainder obtained on dividing n by 9. For example, $f(263) = 2$.
   (a) Let n be a three-digit number and m be the sum of its digits. Show that $f(m) = f(n)$.
   (b) Show that $f(n_1 n_2) = f(f(n_1) \cdot f(n_2))$ where n_1, n_2 are any two positive three-digit integers.

10. Show that it is not possible to have a triangle with sides a, b and c whose medians have lengths $\frac{2}{3}a$, $\frac{2}{3}b$ and $\frac{4}{5}c$.

11. Let

$$P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

be a polynomial with integer coefficients, such that $P(0)$ and $P(1)$ are odd integers. Show that:
   (a) $P(x)$ does not have any even integer as root.
   (b) $P(x)$ does not have any odd integer as root.

12. Let $N = \{1, 2, \ldots, n\}$ be a set of elements called voters. Let $\mathcal{C} = \{S : S \subseteq N\}$ be the set of all subsets of N. Members of $\mathcal{C}$ are called coalitions. Let f be a function from $\mathcal{C}$ to $\{0, 1\}$. A coalition $S \subseteq N$ is said to be *winning* if $f(S) = 1$; it is said to be a *losing* coalition if $f(S) = 0$. A pair $\langle N, f \rangle$ as above is called a voting game if the following conditions hold.
   (a) N is a winning coalition.
   (b) The empty set $\emptyset$ is a losing coalition.
   (c) If S is a winning coalition and $S \subseteq S'$, then S' is also winning.
   (d) If both S and S' are winning coalitions, then $S \cap S' \neq \emptyset$, i.e., S and S' have a common voter.

   Show that the maximum number of winning coalitions of a voting game is 2^{n-1}. Find a voting game for which the number of winning coalitions is 2^{n-1}.

13. Suppose f is a real-valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$. Suppose, moreover, that f satisfies $f'(x) = 1/(x^2 + f^2(x))$. Show that $f(x) \leq 1 + \pi/4$ for every $x \geq 1$.

14. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at some point makes an angle θ with the X-axis, show that the equation of the normal is $y \cos\theta - x \sin\theta = a \cos 2\theta$.

15. Suppose that a is an irrational number.
   (a) If there is a real number b such that both $(a+b)$ and ab are rational numbers, show that a is a quadratic surd. (a is a quadratic surd if it is of the form $r + \sqrt{s}$ or $r - \sqrt{s}$ for some rationals r and s, where s is not the square of a rational number).
   (b) Show that there are two real numbers b_1 and b_2 such that

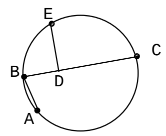
(i) $a + b_1$ is rational but ab_1 is irrational.
(ii) $a + b_2$ is irrational but ab_2 is rational. (Hint: Consider the two cases, where a is a quadratic surd and a is not a quadratic surd, separately).

16. Let A, B, and C be three points on a circle of radius 1.
   (a) Show that the area of the triangle ABC equals

$$\frac{1}{2}\left(\sin(2\angle ABC) + \sin(2\angle BCA) + \sin(2\angle CAB)\right).$$

   (b) Suppose that the magnitude of $\angle ABC$ is fixed. Then show that the area of the triangle ABC is maximized when $\angle BCA = \angle CAB$.
   (c) Hence or otherwise show that the area of the triangle ABC is the maximum when the triangle is equilateral.

17. In the given figure, E is the midpoint of the arc $ABEC$ and ED is perpendicular to the chord BC at D. If the length of the chord AB is l_1, and that of BD is l_2, determine the length of DC in terms of l_1 and l_2.



18. (a) Let $f(x) = x - xe^{-1/x}, x > 0$. Show that $f(x)$ is an increasing function on $(0, \infty)$, and $\lim_{x\to\infty} f(x) = 1$.
   (b) Using part (a) and calculus, sketch the graphs of $y = x-1$, $y = x$, $y = x + 1$, and $y = xe^{-1/|x|}$ for $-\infty < x < \infty$ using the same X and Y axes.

19. For any integer n greater than 1, show that

$$2^n < \binom{2n}{n} < \frac{2^n}{\prod_{i=0}^{n-1}(1 - \frac{i}{n})}.$$

20. Show that there exists a positive real number $x \neq 2$ such that $\log_2 x = x/2$. Hence obtain the set of real numbers c such that

$$\frac{\log_2 x}{x} = c$$

has only one real solution.

21. Find a four digit number M such that the number $N = 4 \times M$ has the following properties.
   (a) N is also a four digit number.
   (b) N has the same digits as in M but in the reverse order.

22. Consider a function f on nonnegative integers such that $f(0) = 1$, $f(1) = 0$ and $f(n) + f(n-1) = nf(n-1) + (n-1)f(n-2)$ for $n \geq 2$.

Show that

$$\frac{f(n)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

23. Of all triangles with a given perimeter, find the triangle with the maximum area. Justify your answer.

24. A 40 feet high screen is put on a vertical wall 10 feet above your eye-level. How far should you stand to maximize the angle subtended by the screen (from top to bottom) at your eye?

25. Study the derivatives of the function

$$y = \sqrt{x^3 - 4x}$$

and sketch its graph on the real line.

26. Suppose P and Q are the centres of two disjoint circles C_1 and C_2 respectively, such that P lies outside C_2 and Q lies outside C_1. Two tangents are drawn from the point P to the circle C_2, which intersect the circle C_1 at points A and B. Similarly, two tangents are drawn from the point Q to the circle C_1, which intersect the circle C_2 at points M and N. Show that $AB = MN$.

27. Evaluate: $\displaystyle\lim_{n\to\infty} \frac{1}{2n} \log \binom{2n}{n}$.

28. Consider the equation $x^5 + x = 10$. Show that
    (a) the equation has only one real root;
    (b) this root lies between 1 and 2;
    (c) this root must be irrational.

29. In how many ways can you divide the set of eight numbers $\{2, 3, \ldots, 9\}$ into 4 pairs such that no pair of numbers has g.c.d. equal to 2?

30. Suppose S is the set of all positive integers. For $a, b \in S$, define

$$a * b = \frac{\text{l.c.m.}(a, b)}{\text{g.c.d.}(a, b)}$$

For example, $8 * 12 = 6$.

Show that ***exactly two*** of the following three properties are satisfied :
    (a) If $a, b \in S$ then $a * b \in S$.
    (b) $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.
    (c) There exists an element $i \in S$ such that $a * i = a$ for all $a \in S$.