



University of Hyderabad,
Entrance Examination, 2004
Ph.D. (Mathematics/Applied Mathematics)

Hall Ticket No.

--	--	--	--	--	--	--

Time: 2 hours

Max. Marks: 75

Part A: 25

Part B: 50

Instructions

1. Calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries 1 mark and each wrong answer carries $-\frac{1}{4}$ mark. So do not gamble. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 50 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Use a separate booklet for Part B.

Answer Part A by circling the correct letter in the array below:

1	a	b	c	d	e
2	a	b	c	d	e
3	a	b	c	d	e
4	a	b	c	d	e
5	a	b	c	d	e

6	a	b	c	d	e
7	a	b	c	d	e
8	a	b	c	d	e
9	a	b	c	d	e
10	a	b	c	d	e

11	a	b	c	d	e
12	a	b	c	d	e
13	a	b	c	d	e
14	a	b	c	d	e
15	a	b	c	d	e

16	a	b	c	d	e
17	a	b	c	d	e
18	a	b	c	d	e
19	a	b	c	d	e
20	a	b	c	d	e

21	a	b	c	d	e
22	a	b	c	d	e
23	a	b	c	d	e
24	a	b	c	d	e
25	a	b	c	d	e



PART A (25 marks)

This part contains 25 questions. Each correct answer carries **1 mark** while a wrong answer carries **minus quarter mark**.

- (1) The set $\{x \in \mathbb{R} : |x + 1| = |x| + 1\}$ is same as
 (a) $\{x \in \mathbb{R} : x > 0\}$.
 (b) $\{x \in \mathbb{R} : x \geq 0\}$.
 (c) whole \mathbb{R} .
 (d) the empty set.
 (e) none of the above.
- (2) The range of the function $f(x) = \frac{x}{\sqrt{x^2 + 1}}$, $x \in \mathbb{R}$ is
 (a) $[-1, 1]$. (b) $[-1, 1)$. (c) $(-1, 1]$. (d) $(-1, 1)$. (e) none of these.
- (3) The set $\{z \in \mathbb{C} : z^3 \text{ is not a real number}\}$ is
 (a) a bounded open connected subset of \mathbb{C} .
 (b) a bounded open disconnected subset of \mathbb{C} .
 (c) an unbounded open disconnected subset of \mathbb{C} .
 (d) an unbounded connected but not open subset of \mathbb{C} .
 (e) none of the above.
- (4) Consider the sequence $\{a_n\}$ of real numbers where $a_1 > 1$ and $a_{n+1} = 2 - \frac{1}{a_n}$, $n \geq 1$. Then the sequence $\{a_n\}$ is
 (a) bounded but not monotone.
 (b) not bounded but monotone
 (c) both bounded and monotone.
 (d) neither bounded nor monotone.
 (e) none of the above.
- (5) Let f be a real valued function defined on $[a, b]$ such that f is differentiable on (a, b) and both $f(a+)$, $f(b-)$ exist. Then
 (a) there is a $c \in (a, b)$ such that $(f(b-) - f(a+)) = f'(c)(b - a)$.
 (b) there is a $c \in (a, b)$ such that $(f(b-) - f(a)) = f'(c)(b - a)$.
 (c) there is a $c \in (a, b)$ such that $(f(b) - f(a+)) = f'(c)(b - a)$.
 (d) there is a $c \in (a, b)$ such that $(f(b) - f(a)) = f'(c)(b - a)$.
 (e) there is no $c \in (a, b)$ satisfying any of the above statements.

(6) Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{xy^2}{x^2 + y^4}, & \text{if } x \neq 0 \text{ and } y \in \mathbb{R} \\ 0, & \text{if } x = 0 \text{ and } y \in \mathbb{R}. \end{cases} \text{ Then}$$

- (a) f is continuous but not differentiable at $(0, 0)$.
 (b) f is differentiable at $(0, 0)$.
 (c) f has all first order partial derivatives at $(0, 0)$.
 (d) f does not have all first order partial derivatives at $(0, 0)$.
 (e) none of the above.
- (7) The function $f(z) = |z|^4$, $z \in \mathbb{C}$ is
 (a) not continuous at 0.
 (b) continuous but not differentiable at 0.
 (c) differentiable but not analytic at 0.
 (d) analytic at 0.
 (e) none of the above.
- (8) A harmonic conjugate $v(x, y)$ of the function $u(x, y) = x^3 - 3xy^2$ on \mathbb{C} is
 (a) $y^3 - 3x^3$.
 (b) $3x^2y - y^3$.
 (c) $x^3 - 3xy^2$.
 (d) $x^3 - 3x^2y$.
 (e) none of the above.
- (9) The residue of $f(z) = \frac{e^{z^3}}{(z-1)^3}$ at $z = 1$ is
 (a) $\frac{9e}{2}$. (b) $\frac{15e}{2}$. (c) $\frac{21e}{2}$. (d) $\frac{27e}{2}$. (e) none of these.
- (10) The value of $3^{37} \pmod{481}$ is
 (a) 3. (b) 36. (c) 37. (d) 13. (e) none of these.
- (11) Let C_{12} be a cyclic group generated by a . The order of the element $(a^7, a^6) \in C_{12} \times C_{12}$ is
 (a) 2. (b) 5. (c) 6. (d) 12. (e) none of these.
- (12) The number of Sylow 7 - subgroups in a group of order 392 is
 (a) 2 or 3.
 (b) 3 or 7.
 (c) 1 or 8.
 (d) 2 or 7.
 (e) none of the above.

- (13) Consider the quotient ring $\mathbb{Z}/95\mathbb{Z}$, where \mathbb{Z} is the ring of integers. The number of elements $x \in \mathbb{Z}/95\mathbb{Z}$ such that $x^2 = \bar{1}$ is
 (a) 1. (b) 2. (c) 3. (d) 4. (e) none of these.
- (14) The number of proper subfields of the field $\mathbb{F}_{2^{35}}$ (finite field with 2^{35} elements) is
 (a) 1. (b) 2. (c) 3. (d) 4. (e) none of these.
- (15) Denote by \mathbb{Z}_p , the field of integers modulo p , where p is a prime. In the vector space \mathbb{Z}_p the number of distinct one dimensional subspaces is
 (a) $p + 2$. (b) $p + 1$. (c) $p - 1$. (d) $p - 2$. (e) none of these.
- (16) If A is a 4×4 matrix with characteristic polynomial $(x^2 - 1)^2$ and minimal polynomial $(x^2 - 1)$, then its Jordan canonical form is

$$\begin{array}{l}
 \text{(a)} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 \text{(c)} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{(d)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
 \text{(e) none of the above.}
 \end{array}$$

- (17) Let X and Y be two topological spaces and $f : X \rightarrow Y$ be a continuous surjection. Then
 (a) V is an open set in X implies $f(V)$ is open in Y .
 (b) F is a closed set in X implies $f(F)$ is closed in Y .
 (c) D is dense in X implies $f(D)$ is dense in Y .
 (d) A is an infinite subset of X implies $f(A)$ is an infinite subset of Y .
 (e) none of the above.
- (18) Let m be the Lebesgue measure on \mathbb{R} and E be a Lebesgue measurable subset of \mathbb{R} . For any real a , if we denote $aE = \{ax : x \in E\}$, then
 (a) $m(aE) = m(E)$.
 (b) $m(aE) = am(E)$.
 (c) $m(aE) = |a|m(E)$.
 (d) $m(aE) = a + m(E)$.
 (e) none of the above.

- (19) On the vector space $C^1[0, 1]$ of all real-valued continuously differentiable functions defined on $[0, 1]$, consider the following norms $\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)|$, $\|f\|_1 = \int_0^1 |f(x)| dx$, $\|f\|_2 = \|f\|_\infty + \|f'\|_\infty$, and $\|f\|_3 = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Then
- $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent.
 - $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
 - $\|\cdot\|_1$ and $\|\cdot\|_3$ are equivalent.
 - $\|\cdot\|_\infty$ and $\|\cdot\|_3$ are equivalent.
 - $\|\cdot\|_2$ and $\|\cdot\|_3$ are equivalent.
- (20) Among the following statements, which one does not characterize a finite-dimensional normed linear space X ?
- every linear subspace of X is closed.
 - the closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is compact.
 - every bounded sequence in X has a convergent subsequence.
 - every continuous functional on a linear subspace of X has a unique Hahn-Banach extension.
 - every functional on X is continuous.
- (21) Consider the linear subspace $Y = \{(x, y, z) \in \mathbb{R}^3 : x = y, z = 0\}$ of \mathbb{R}^3 with the norm $\|(x, y, z)\| = |x| + |y| + |z|$ and $f_0 : Y \rightarrow \mathbb{R}$ be a linear functional given $f_0(x, y, z) = 2x$, where $(x, y, z) \in Y$. Then
- $f_1 : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f_1(x, y, z) = 2x + z$ is a Hahn-Banach extension of f_0 .
 - $f_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f_2(x, y, z) = 2y + z$ is a Hahn-Banach extension of f_0 .
 - $f_3 : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f_3(x, y, z) = x - y$ is a Hahn-Banach extension of f_0 .
 - $f_4 : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f_4(x, y, z) = x + y - z$ is a Hahn-Banach extension of f_0 .
 - none of the above is a Hahn-Banach extension of f_0 .
- (22) The general solution of the fourth order differential equation $y^{(iv)} - 5y''' + 6y'' + 4y' - 8y = 0$ is
- $y = (c_1 + c_2x + c_3x^2)e^{2x} + c_4e^{-x}$.
 - $y = c_1e^{2x} + c_2e^{-x}$.
 - $y = c_1 + c_2xe^x + c_3x^2e^{2x} + c_4e^{-x}$.
 - $y = (c_1 + c_2x + c_3x^2)e^{-x} + c_4e^{2x}$.
 - none of the above.

(23) The critical point $(0, 0)$ of the following system

$$\frac{dx}{dt} = (y + 1)^2 - \cos x,$$

$$\frac{dy}{dt} = \sin(x + y),$$

is

- (a) a saddle point.
 - (b) an unstable node.
 - (c) a stable node.
 - (d) an unstable spiral.
 - (e) a stable spiral.
- (24) The extremum of the functional $J[y] = \int_0^1 (y^2 + 2yy')dx$, satisfying $y(0) = 1$ and $y(1) = 0$
- (a) is a straight line
 - (b) is a parabola.
 - (c) is an ellipse
 - (d) does not exist.
 - (e) none of the above.
- (25) The equation $u_{xx} + xu_{xy} + yu_{yy} - xyu_x = 0$ is hyperbolic for
- (a) $y > \frac{x^2}{4}$.
 - (b) $y < \frac{x^2}{4}$.
 - (c) $y = \frac{x^2}{4}$.
 - (d) $y = x$.
 - (e) none of the above.

Part B (50 Marks)

There are 15 questions in this part. Each question carries 5 marks. Attempt **any 10 questions**. The answers should be written in the separate booklet given.

- (1) Consider the sequence $\{f_n\}$ of real valued functions, where $f_n(x) = \frac{1}{1 + nx}$, $x \in \mathbb{R}$. Answer the following questions with proper justification.
- (a) Is $\{f_n\}$ pointwise convergent on $[0, 1]$?
 - (b) Is $\{f_n\}$ uniformly convergent on $[0, 1]$?
 - (c) Is $\{f_n\}$ uniformly bounded on $[0, 1]$?
 - (d) Is $\{f_n\}$ bounded in $L^1[0, 1]$?
 - (e) Is $\{f_n\}$ convergent on $L^1[0, 1]$?

- (2) If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then show that f is either continuous at each point of \mathbb{R} or not continuous at any point of \mathbb{R} .
- (3) Let $a > 0$ and $\{x_n\}$ be a sequence such that $x_1 = 1, x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ for $n = 1, 2, \dots$. Show that $\lim_{n \rightarrow \infty} x_n = \sqrt{a}$.
- (4) Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by $f(x, y, z) = x^2y + e^x + z$ and $\underline{a} = (0, 1, -1)$. Using implicit function theorem, find the implicit function $g(y, z)$ in a neighbourhood of $(1, -1)$ in \mathbb{R}^2 and compute $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$ at $(1, -1)$.
- (5) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f has derivatives of all orders on \mathbb{R} and $f^k(0) = 0$ for all integers $k \geq 1$.

- (6) Determine the number of roots of the polynomial $3z^7 + 5z - 1$ inside the circles $|z| = 1, |z| = 2$ and in the annulus $1 < |z| < 2$.
- (7) Using Residue theorem, find the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2 + 1)^2} dx.$$

- (8) Let M_n^λ stand for the $n \times n$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdot & 0 & 0 \\ 0 & \lambda & 1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \cdot & 0 & \lambda \end{pmatrix}.$$

What will be the characteristic polynomial and the minimal polynomial of the 9×9 matrix whose diagonal blocks are $M_3^2, M_2^2, M_1^2, M_2^{-2}, M_1^{-2}$. Justify your answer.

- (9) Let T be a non-zero $n \times n$ matrix. If T commutes with all $n \times n$ matrices, then show that T is a scalar multiple of the identity matrix.
- (10) Show that a group of order 30 must contain a subgroup of order 15. Is it necessarily normal?
- (11) Let α be a root of $x^3 + x + 1$ and β be a root of $x^3 + x + 3$. Show that it is not possible that $\alpha \in \mathbb{Q}(\beta)$.
- (12) Suppose the density of a subset A of \mathbb{N} , the set of natural numbers is defined as $\lim_{n \rightarrow \infty} \frac{a_n}{n}$, if it exists, where a_n is the cardinality of the set $A \cap \{1, 2, \dots, n\}$. The density may or may not exist for a subset A of \mathbb{N} . Clearly if it exists, then it is a number in $[0, 1]$.

- (a) If m is a fixed positive integer and $A = \{km : k \in \mathbb{N}\}$, then find the density of A .
- (b) What is the density of the set $\{2^n : n \in \mathbb{N}\}$?
- (13) Consider the Banach spaces $C[0, 1]$ of all real-valued continuous functions on $[0, 1]$ with the supremum norm $\|f\|_\infty$ and $C^1[0, 1]$ of all real valued continuously differentiable functions on $[0, 1]$ with the norm $\|f\|_\infty + \|f'\|_\infty$. For $f \in C[0, 1]$, define \tilde{f} on $[0, 1]$ by $\tilde{f}(x) = \int_0^x f(t)dt$, $0 \leq x \leq 1$, which belongs to $C^1[0, 1]$. Define $T : C[0, 1] \rightarrow C^1[0, 1]$ by $Tf = \tilde{f}$. Show that T is a bounded linear map. Find the value of $\|T\|$.
- (14) Find the characteristic values and characteristic functions of

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) + \frac{\lambda}{x}y = 0, \quad y(1) = 0, \quad y(e^\pi) = 0,$$

where λ is a non-negative number.

- (15) Find the integral surface for the partial differential equation $z(xp - yq) = y^2 - x^2$, passing through the initial data curve $(2s, s, s)$.

