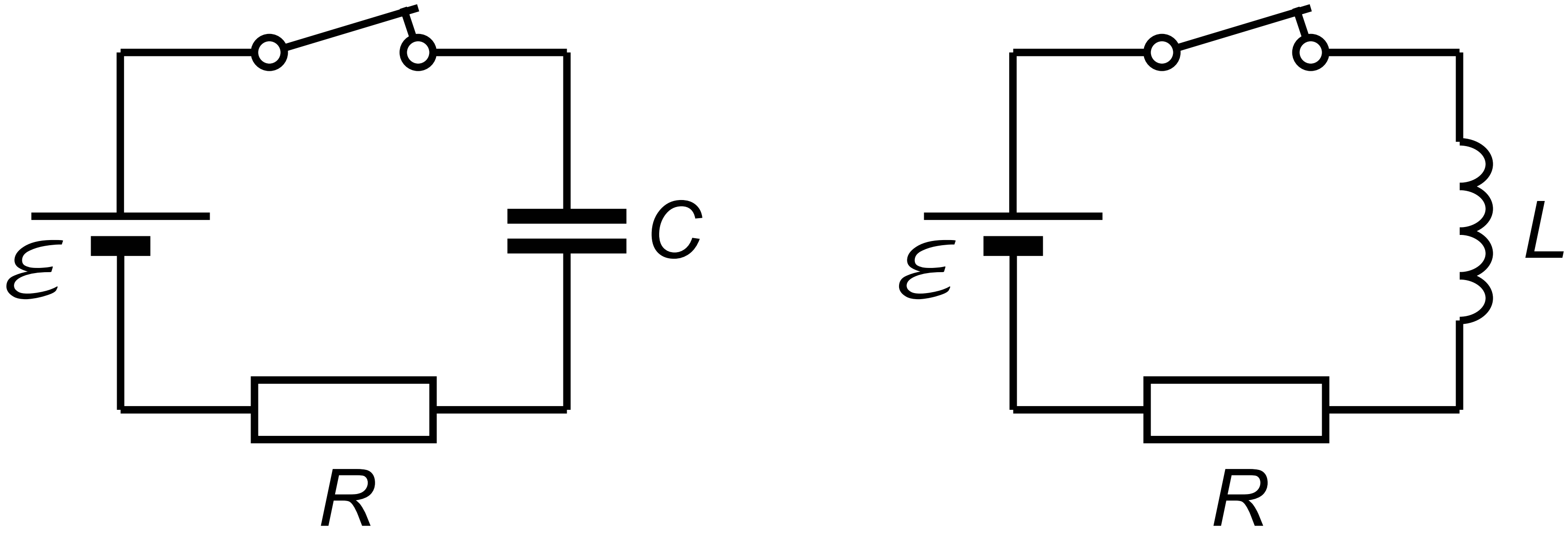
2. CAPACITANCE AND INDUCTANCE IN D.C. CIRCUITS

Main things to learn

- Transients in *CR* and *LR* circuits
- Time constant



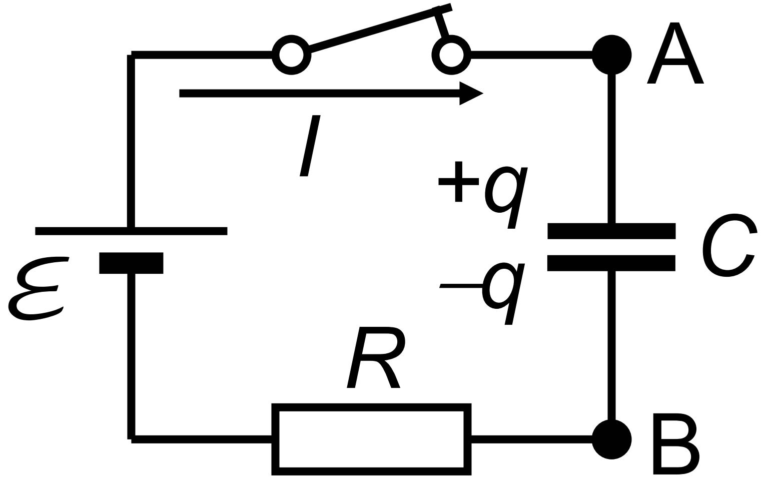
In a steady d.c. circuit: *C* is equivalent to the open circuit

L is equivalent to the short circuit

Our aim is to study the **transient processes** in *CR* and *LR* circuits

What will be happening in such circuits just after the switch is turned on / off ?

TRANSIENT IN A *CR* CIRCUIT



q - charge at the plates of the capacitor

$\mathcal{E}$ - electromotive force in the battery

I - current in the direction of the arrow

U_{AB} - potential difference between A and B

Before the switch is turned on: $q = 0$, $U_{AB} = 0$, $I = 0$

Long time after (established): $U_{AB} = \mathcal{E}$, $q = C\cdot\mathcal{E}$, $I = 0$

After the switch is turned on, a transient from $q = 0$ to $q = C\cdot\mathcal{E}$ takes place

How to analyse the transient? - The transient is **slow**, therefore

at **every moment of time** t , the 2nd Kirchhoff's law is valid

$$U_{AB}(t) - \mathcal{E} = - I(t)\cdot R$$

$$q(t) = C\cdot U_{AB}(t)$$

Relation between charge and current (note sign): $I(t) = \dfrac{dq(t)}{dt}$

These three equations result in a differential equation

$$\frac{q(t)}{C} - \mathcal{E} = -R\frac{dq(t)}{dt}$$

$$\frac{dq(t)}{dt} = -\frac{1}{RC}(q(t) - C\mathcal{E})$$

We denote

$$y(t) = q(t) - C\mathcal{E}$$

$$\tau = RC$$

$$\frac{dy}{dt} = -\frac{y}{\tau} \quad \therefore \quad y = y_0 \exp(-t/\tau)$$

(y_0 - value of y at $t = 0$: $\quad y_0 = -C\mathcal{E}$)

$$q(t) = C\mathcal{E}[1 - \exp(-t/RC)] = C\mathcal{E}[1 - \exp(-t/\tau)]$$

$$U_{\mathrm{AB}} = \mathcal{E}[1 - \exp(-t/RC)] = \mathcal{E}[1 - \exp(-t/\tau)]$$

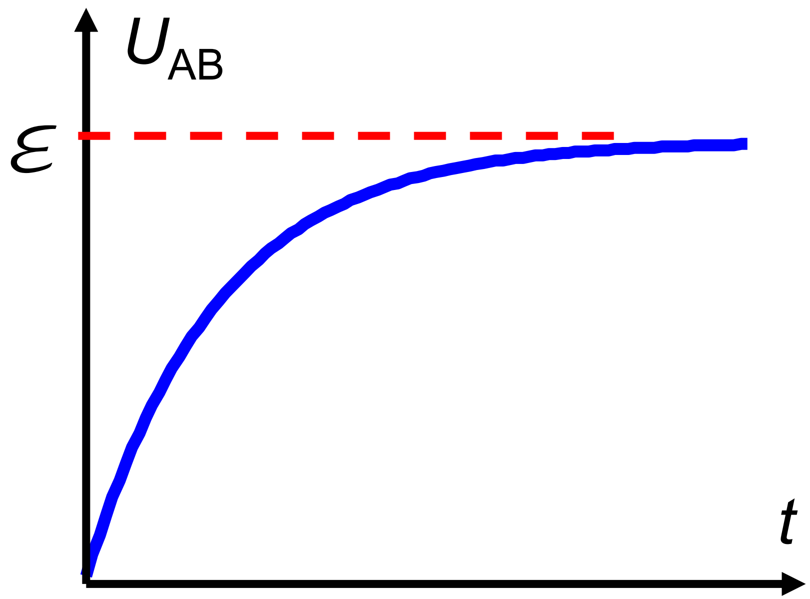
$$I(t) = \frac{\mathcal{E}}{R}\exp(-t/RC) = \frac{\mathcal{E}}{R}\exp(-t/\tau)$$

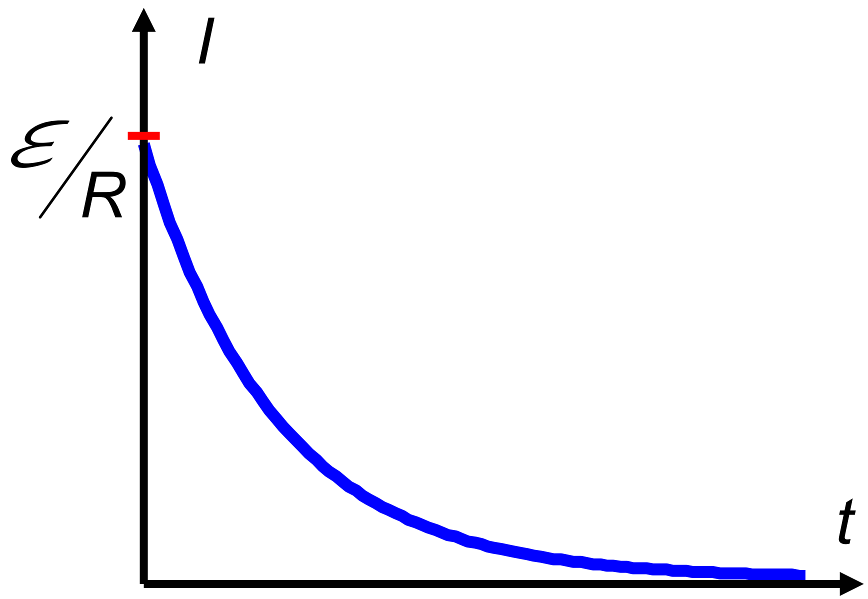
$$\boxed{\tau = RC}$$

- time constant

t	τ	2τ	3τ	4τ
exp $(-t/\tau)$	0.37	0.14	0.05	0.02

ANALYSIS OF OBTAINED RESULTS





At $t = 0$: $\quad q = 0; \;\; U_{AB} = 0; \;\; I = \mathcal{E}/R$

At $t = \infty$: $\quad q = C\mathcal{E}; \;\; U_{AB} = \mathcal{E}; \;\; I = 0$

Though the capacitor provides the open circuit,

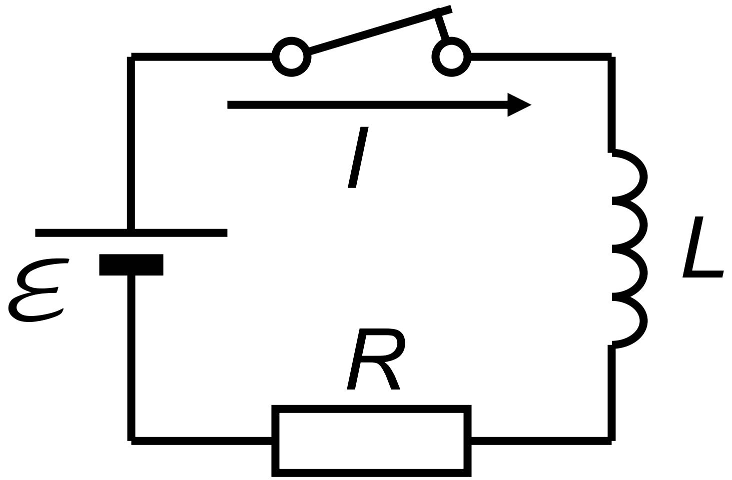
the current at t = 0 is such as if it is a short circuit !

This happens because at t = 0, U_{AB} = 0

$$\tau = RC$$

R = 1 MΩ and C = 1 μF : τ = 1 sec

TRANSIENT IN A *LR* CIRCUIT



L - self-inductance of the coil

$\mathcal{E}_i$ -EMF of the electromagnetic induction

Φ - flux of magnetic field through the coil

$$\Phi = L \cdot I$$

Reminder: Electromagnetic induction in the coil

$$\mathcal{E}_i = -\frac{d\Phi(t)}{dt} = -L\frac{dI(t)}{dt}$$

Before the switch is turned on: $I = 0$

Long time after (established): $I = \mathcal{E}/R$

After the switch is turned on, a transient from $I = 0$ to $I = \mathcal{E}/R$ takes place

At **every moment of time** t , the 2nd Kirchhoff's law is valid

$$\boxed{\mathcal{E} + \mathcal{E}_i = I \cdot R}$$

A differential equation is obtained

$$\mathcal{E} - L\frac{dI(t)}{dt} = I \cdot R$$

$$\frac{dI(t)}{dt} = -\frac{R}{L}\left(I(t) - \frac{\mathcal{E}}{R}\right)$$

We denote

$$y(t) = I(t) - \frac{\mathcal{E}}{R} \quad \text{and} \quad \tau = L/R$$

$$\frac{dy}{dt} = -\frac{y}{\tau} \quad \therefore \quad y = y_0 \exp(-t/\tau)$$

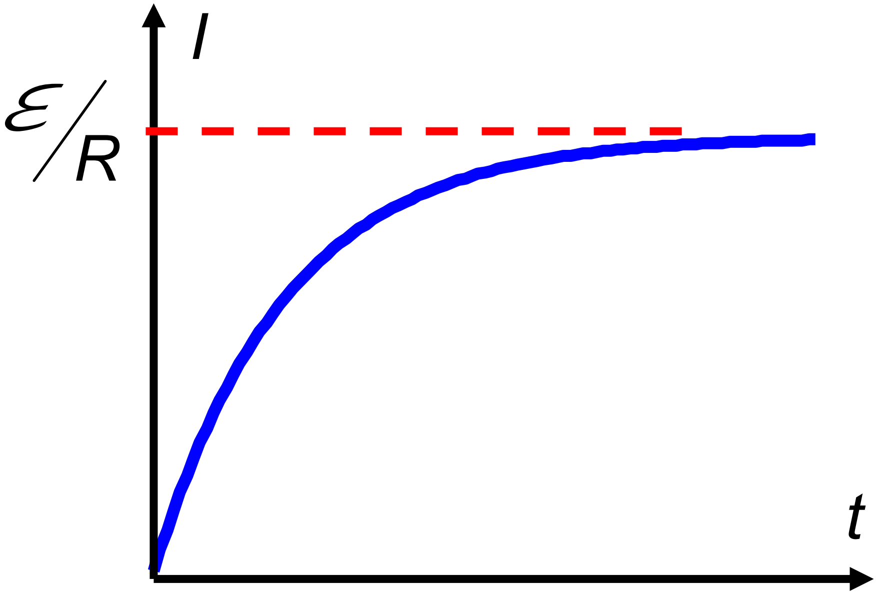
$$(y_0 \text{ - value of } y \text{ at } t = 0: \quad y_0 = -\frac{\mathcal{E}}{R})$$

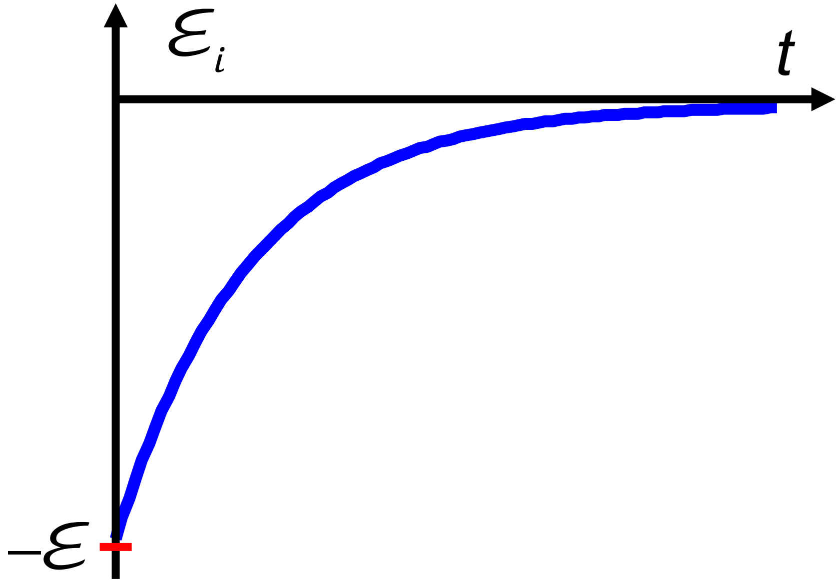
$$I(t) = \frac{\mathcal{E}}{R}\left[1 - \exp\left(-\frac{t \cdot R}{L}\right)\right] = \frac{\mathcal{E}}{R}\left[1 - \exp(-t/\tau)\right]$$

$$\mathcal{E}_i(t) = \mathcal{E}\exp\left(-\frac{t \cdot R}{L}\right) = \mathcal{E}\exp(-t/\tau)$$

$\boxed{\tau = L/R}$ - time constant for an LR circuit

ANALYSIS OF OBTAINED RESULTS





$$\text{At } t = 0: \quad I = 0; \quad \mathcal{E}_i = -\mathcal{E}$$

$$\text{At } t = \infty: \quad I = \frac{\mathcal{E}}{R}; \quad \mathcal{E}_i = 0$$

Though the coil provides the short circuit (its resistance is almost zero),

the current at t = 0 is such as if it is an open circuit

At t = 0, the external EMF is **fully** compensated by the EMF of self-inductance

$$\tau = L/R$$

$\boxed{R = 1\ \Omega \text{ and } L = 1 \text{ henry}: \quad \tau = 1 \text{ sec}}$

CONCLUSION

If a circuit includes a capacitance or an inductance,

final values of currents or charges **cannot** establish **instantaneously**

There are always **transient processes** which are characterised

by time constants

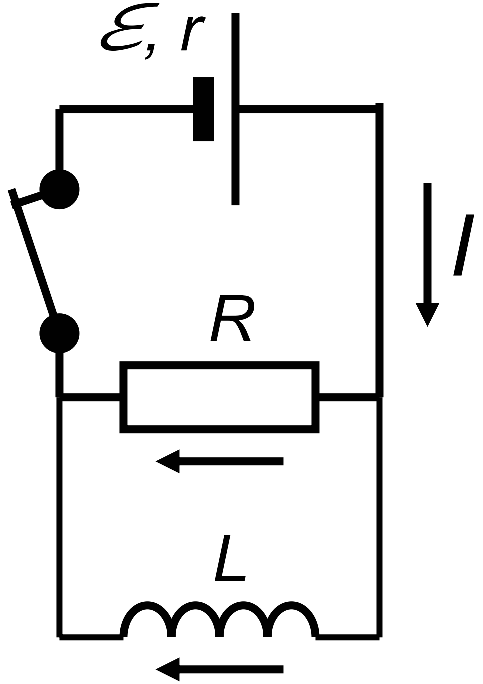
For an *RC* circuit
time constant

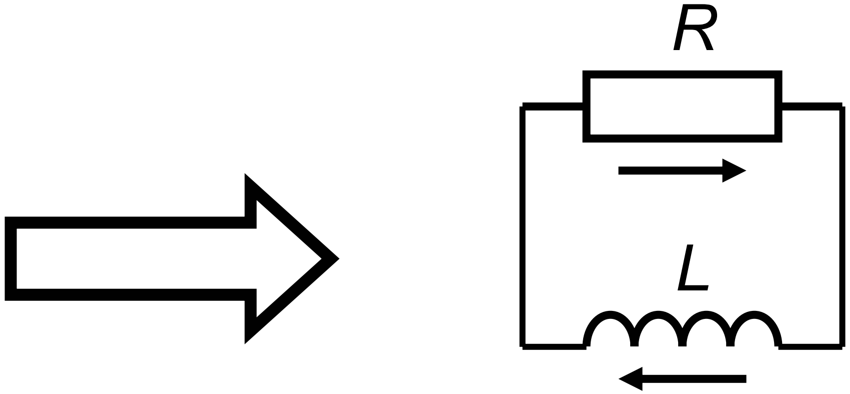
$$\boxed{\tau = RC}$$

For an *LR* circuit
time constant

$$\boxed{\tau = L/R}$$

ENERGY STORED IN A SELF-INDUCTANCE (*)





After the switch is turned off, the current is driven by the EMF $\mathcal{E}_i$ which is due to the self-inductance L

$$\mathcal{E}_i = -L\frac{\Delta I}{\Delta t} \quad (I \text{ changes from } I_0 \text{ to } 0)$$

Work done by the self-inductance EMF

$$A = \sum \mathcal{E}_i \cdot I\Delta t = -L\sum I\Delta I = -\sum \Phi\Delta I = \frac{\Phi \cdot I_0}{2} = \frac{L \cdot I_0^{\,2}}{2}$$

This work was done due to energy W which was accumulated in the coil

$$W = \frac{L \cdot I^2}{2}$$ - energy stored in a self-inductance

ENERGY OF MAGNETIC FIELD (*)

$$W = \frac{L \cdot I^2}{2} = \frac{1}{2} \cdot \mu_0 \cdot n^2 \cdot V \cdot I^2$$

$$B = \mu_0 \cdot n \cdot I$$

$$W = \frac{1}{2\mu_0} \cdot B^2 \cdot V$$ - energy of magnetic field in the volume V

$$w = \frac{B^2}{2\mu_0}$$ - energy density of magnetic field

It is similar to $w = \frac{\varepsilon_0 E^2}{2}$ - energy density of electric field

This similarity reflects the fact that both electric and magnetic fields are manifestations of the electromagnetic field