Syllabus for the M. Math. Selection Test–2006
TEST CODE MM

Open sets, closed sets and compact sets in $\mathbf{R}^n$;

Convergence and divergence of sequences and series;

Continuity, uniform continuity, differentiability, mean-value theorem;

Pointwise and uniform convergence of sequences and series of functions, Taylor expansions, power series;

Integral calculus of one variable : Riemann integration, Fundamental theorem of calculus, change of variables;

Directional and total derivatives, Jacobians, chain rule;

Maxima and minima of functions of one and several variables;

Elementary topological notions for metric spaces : compactnes, connectedness, completeness;

Elements of ordinary differential equations.

Equivalence relations and partitions;

Primes and divisibility;

Groups : subgroups, products, quotients, homomorphisms, Lagrange's theorem, Sylow's theorems;

Commutative rings : Ideals, prime and maximal ideals, quotients, congruence arithmetic, integral domains, field of fractions, principal ideal domains, unique factorization domains, polynomial rings;

Fields : field extensions, roots and factorization of polynomials, finite fields;

Vector spaces: subspaces, basis, dimension, direct sum, quotient spaces;

Matrices : systems of linear equations, determinants, eigenvalues and eigenvectors, diagonalization, triangular forms;

Linear transformations and their representation as matrices, kernel and image, rank;

Inner product spaces, orthogonality and quadratic forms, conics and quadrics.

Sample Questions for the Selection Test

Notation : $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers, rational numbers, integers and natural numbers respectively.

(1) Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n\geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n\to\infty} f(x_n)$ exists.

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \max \{| x |, | y |\} .$$

Show that f is a uniformly continuous function.

(3) A map $f : \mathbb{R} \to \mathbb{R}$ is called open if $f(A)$ is open for every open subset A of $\mathbb{R}$. Show that every continuous open map of $\mathbb{R}$ into itself is monotonic.

(4) Let $S = \{(x_1, x_2, ...x_n) \in \mathbb{R}^n : \sum | x_i |^2 = 1\}$. Let

$$A = \{(y_1, y_2, ...y_n) \in \mathbb{R}^n : \sum \frac{y_i}{i} = 0\}.$$

Show that the set $S + A = \{x + y : x \in S \ , \ y \in A\}$ is a closed subset of $\mathbb{R}^n$.

(5) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $f : [0, 1] \to \mathbb{C}$ be continuous with $f(0) = 0, \ f(1) = 2$. Show that there exists at least one t_0 in $[0, 1]$ such that $f(t_0)$ is in $\mathbb{T}$.

(6) Let f be a continuous function on $[0, 1]$. Evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx$$

(7) Let $N > 0$ and let $f : [0, 1] \to [0, 1]$ be denoted by $f(x) = 1$ if $x = 1/i$ for some integer $i \leq N$ and $f(x) = 0$ for all other values of x. Show that f is Riemann integrable.

(8) Let $f : (0, 1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational ,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ with } m, n \text{ relatively prime} \end{cases}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x > \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Show that $g \circ f$ is not Riemann integrable.

(9) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$. Define

$$f_n(x) = f(nx), \text{ for } x \in \mathbb{R} \text{ and } n = 1, 2, 3, \ldots$$

Suppose that $\{f_n\}$ is equicontinuous on $[0, 1]$, that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in [0, 1]$, $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \varepsilon$ for all n. Show that $f(x) = 0$ for all $x \in [0, 1]$.

(10) Find the most general curve in $\mathbb{R}^2$ whose normal at each point passes through $(0, 0)$. Find the particular curve through $(2, 3)$.

(11) Find the maximum value of the function

$$f(x, y, z) = s(s - x)(s - y)(s - z),$$

where $s > 0$ is a given constant under the condition

$$x + y + z - 2s = 0,$$

and where x, y, z are restricted by the inequalities

$$x \geq 0, y \geq 0, z \geq 0,$$

$$x + y \geq z, x + z \geq y, y + z \geq x.$$

(12) Let (X, d) be a compact metric space and $f : X \to X$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is onto.

(13) Let ω be an n-th root of unity such that $\omega^m \neq 1$ for any positive integer $m < n$. Show that $(1 - \omega)...(1 - \omega^{n-1}) = n$ [Hint : Consider the polynomial $z^n - 1$].

Hence deduce the following : if $A_1, A_2, ..., A_n$ are the vertices of a regular n-gon inscribed in a unit circle, prove that

$$l(A_1A_2)l(A_1A_3)...l(A_1A_n) = n,$$

where $l(AB)$ denotes the length of a line segment AB.

(14) Let $f(x)$ be a non-constant polynomial with integer coefficients. Show that the set $S = \{f(n) | n \in \mathbb{N}\}$ has infinitely many composite numbers.

(15) Let G be any group. Prove that any subgroup H of finite index n in G contains a normal subgroup of index dividing $n!$.
Hint : Consider the homomorphism from G to the group of permutations of the set of left cosets of H in G.

(16) Let G be a nonabelian group of order 55. How many subgroups of order 11 does it have? Using this information or otherwise compute the number of subgroups of order 5.

(17) Let $n \in \mathbb{N}$ and p be a prime number. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_\ell x^\ell$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$, where $a_i, b_j \in \mathbb{Z}/p^n\mathbb{Z}$, for all $0 \leq i \leq \ell$, $0 \leq j \leq m$. Suppose that $fg = 0$. Prove that $a_ib_j = 0$ for all $0 \leq i \leq \ell$, $0 \leq j \leq m$.

(18) Let $a_1, a_2, ..., a_n$ be n distinct integers. Prove that the polynomial $f(x) = (x - a_1)(x - a_2)...(x - a_n) + 1$ is irreducible in $\mathbb{Z}[x]$.

(19) Prove that $x^4 - 10x^2 + 1$ is reducible modulo p for every prime p.

(20) Consider the two fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$, where $\mathbb{Q}$ is the field of rational numbers. Show that they are isomorphic as vector spaces but not isomorphic as fields.

(21) Show that the only field automorphism of $\mathbb{Q}$ is the identity. Using this prove that the only field automorphism of $\mathbb{R}$ is the identity.

(22) Suppose $f \in F[x]$ be an irreducible polynomial of degree 5, where F is a field. Let K be a quadratic field extension of F, that is, $[K : F] = 2$. Prove that f remains irreducible over K.

(23) Let $k[x, y]$ be the polynomial ring in two variables x and y over a field k. Prove that any ideal of the form $I = (x - a, y - b)$ for $a, b \in k$ is a maximal ideal of this ring. What is the vector space dimension (over k) of the quotient space $k[x, y]/I$?

(24) Let A be a $n \times n$ symmetric matrix of rank 1 over the complex numbers $\mathbb{C}$. Show that $A = \alpha\, \boldsymbol{u}\boldsymbol{u}^t$ for some non-zero scalar

$\alpha \in \mathbb{C}$ and a non-zero vector $\boldsymbol{u} \in \mathbb{C}^n$ (where $\boldsymbol{u}^t$ is the transpose of $\boldsymbol{u}$).

(25) Let A be any 2×2 matrix over $\mathbb{C}$ and let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be any polynomial over $\mathbb{C}$. Show that $f(A)$ is a matrix which can be written as $c_0 I + c_1 A$ for some $c_0, c_1 \in \mathbb{C}$, where I is the identity matrix.

(26) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that there is a line L through origin such that $T(L) = L$

(27) Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0$ for all $(i, j) \neq (1, 2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.

Model question paper

Time : 2 hours

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a): Attempt any three questions from each group.
b): Each question carries equal weightage.
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Group A (any three)

(1) Suppose that f is a real-valued continuous function defined on $\mathbb{R}$ and $f(x+1) = f(x)$ for all $x \in \mathbb{R}$.
(a) Show that f is bounded above and below and achieves its maximum and minimum.
(b) Show that f is uniformly continuous on $\mathbb{R}$.
(c) Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.

(2) Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function for each $n \geq 1$, with $|f_n'(x)| \leq 1$ for all n and x. Assume also that $\lim_{n\to\infty} f_n(x) = g(x)$ exists for all $x \in \mathbb{R}$. Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

(3) Suppose that $\{a_k\}_{k=1}^{\infty}$ is a bounded sequence of nonnegative real numbers. Show that $\frac{1}{n}\sum_{k=1}^{n} a_k \to 0$ as $n \to \infty$ if and only if $\frac{1}{n}\sum_{k=1}^{n} a_k^2 \to 0$ as $n \to \infty$.

(4) Consider the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0.$$

Suppose that $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions of this differential equation. Suppose also that there exist $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$ and $y_1(x_1) = 0 = y_1(x_2)$. Show that there is x_3 in $\mathbb{R}$ such that $x_1 < x_3 < x_2$ with $y_2(x_3) = 0$.

(5) Let (X, d) be a metric space. For a closed subset A of X, define the function d_A by

$$d_A(x) := inf\{d(x, y) : y \in A\}.$$

Prove that
(i) $|d_A(x_1) - d_A(x_2)| \leq d(x_1, x_2)$ for all $x_1, x_2 \in X$.
(ii) $d_A(x) = 0$ if and only if $x \in A$.

Group B (any three)

(1) For a prime p, $\mathbb{F}_p (= \mathbb{Z}/p\mathbb{Z})$ denotes the field of integers modulo p. Determine all primes p for which the system of equations

$$8x + 3y = 10$$
$$2x + 6y = -1$$

(i) has no solution in $\mathbb{F}_p$;
(ii) has exactly one solution in $\mathbb{F}_p$;
(iii) has more than one solution in $\mathbb{F}_p$. In case (iii), how many solutions does the system have in $\mathbb{F}_p$?

(2) Let $a, b \in \mathbb{Z}$ and let d be the G.C.D. of a and b. Let D denote the subring of $\mathbb{Q}$ defined by

$$D = \{\frac{r}{d^k} \in \mathbb{Q} \mid r \in \mathbb{Z},\ k \in \mathbb{N} \cup \{0\}\}.$$

Show that $(a^m, b^n)D = D$ for all $m, n \in \mathbb{N}$; where $(a^m, b^n)D$ denotes the ideal $\{a^m\alpha + b^n\beta \mid \alpha, \beta \in D\}$.

(3) A field L is called an algebraic extension of a subfield k if, for each $\alpha \in L$, there exists a nonzero polynomial $f(x) \in k[x]$ such that $f(\alpha) = 0$.

Suppose that k is a field and L is an algebraic extension of k. Show that any subring R of L, such that $k \subseteq R \subseteq L$, is a field.

(4) Let A be an $n \times n$ real matrix such that $A^2 = I$, but $A \neq \pm I$ (where I denotes the $n \times n$-identity matrix). Show that
(i) A has two eigenvalues λ_1, λ_2.
(ii) Every element $x \in \mathbb{R}^n$ can be expressed uniquely as $x_1 + x_2$, where $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$.

(5) Let N be a normal subgroup of a finite group G such that the index $[G : N]$ is relatively prime to $|N|$, where $|N|$ denotes the order of N. Show that there is no other subgroup of G of order $|N|$.