XX-3376 Seat No.________

M. Phil. Examination

April / May – 2003

Mathematics : Paper-IV

Time : **3** Hours] [Total Marks : **75**

1 (a) Give a complete description to show that $\boldsymbol{C(X)}$ is a commutative Banach algebra with identity. ($\boldsymbol{X}$ is a compact, T_2 space). **8**

(b) Define quasi-regular elements in an algebra. If algebra has identity then show that x is quasi-regular if and only if $e - x$ is regular. **8**

OR

1 (a) If $(\boldsymbol{V}, \|\cdot\|)$ is a Banach space over then show that $L(V)$ is a Banach algebra. **8**

(b) Define regular ideal in an algebra. If $I \subset A$ is a proper regular two sided ideal in an algebra A then show that A/I is an algebra with identity. **8**

2 (a) Show that every Banach division algebra is commutative. **8**

(b) Define topologically nilpotent element in a normed algebra. Give an example of a topologically nilpotent element which is not nilpotent. **8**

OR

2 (a) State and prove Gelfand-Mazur theorem using the non-emptiness of $\sigma(\boldsymbol{x})$. **8**

(b) Let A be a Banach algebra with identity c. If $x \in bdy\left(A^{-1}\right)$ then show that x is a two sided topological zero divisor. **8**

3 Attempt any **three** : **18**

(a) Define spectral radius and state spectral radius formula. If A is a Banach algebra then is it true that $\|x\| = \|x\|_{\sigma}, \ \forall x \in A$? Justify your answer.

(b) State and prove spectral mapping theorem for polynomials.

(c) Characterize maximal ideals in the disc algebra $A(D)$.

(d) Define semi-simple Banach algebra with an illustration. Is a semi-simple Banach algebra ? – Explain.

(e) Provide with explanation an example of a commutative Banach algebra A that contains a subalgebra B such that $\Delta(\boldsymbol{A}) \subset \Delta(\boldsymbol{B})$ and $\Delta(A) \neq \Delta(B)$.

4 (a) Determine all the odd primes that can be expressed in the form $x^2 + xy + 5y^2$. **5**

(b) Determine all the positive integers that can be expressed in the form $x^2 - y^2$. **5**

(c) Find the least positive integer that can be represented by $4x^2 + 17xy + 20y^2$. **5**

(d) Prove that there are infinitely many irreducible elements in the integral domain of any quadratic field. **5**

(e) Show that $Q\left(\sqrt{-17}\right)$ does not have unique factorization. **5**

OR

4 (a) Determine all the positive integers that can be expressed in the form $x^2 + 2y^2$. **5**

(b) Determine all the primes that can be represented by $x^2 + 5y^2$. **5**

(c) Calculate the class number $h(-31)$. **5**

(d) Show that $Q(\sqrt{3})$ is Euclidean. Determine algebraic integers α, β is in $Q(\sqrt{3})$ such that 5

$$(1+2\sqrt{3})\alpha + (5+4\sqrt{3})\beta = 1.$$

(e) For $d < 0$, describe the units in $Q(\sqrt{d})$. (*d* is a square-free integer). **5**