

SYLLABUS FOR THE M. MATH. SELECTION TEST–2008
TEST CODE MM

Open sets, closed sets and compact sets in \mathbf{R}^n ;

Convergence and divergence of sequences and series;

Continuity, uniform continuity, differentiability, mean-value theorem;

Pointwise and uniform convergence of sequences and series of functions,
Taylor expansions, power series;

Integral calculus of one variable : Riemann integration, Fundamental
theorem of calculus, Multiple integrals, change of variables;

Directional and total derivatives, Jacobians, chain rule;

Maxima and minima of functions of one and several variables;

Elementary topological notions for metric spaces : compactnes, con-
nectedness, completeness;

Elements of ordinary linear differential equations.

Equivalence relations and partitions;

Primes and divisibility;

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Groups : subgroups, products, quotients, homomorphisms, Lagrange's theorem, Sylow's theorems;

Commutative rings : Ideals, prime and maximal ideals, quotients, congruence arithmetic, integral domains, field of fractions, principal ideal domains, unique factorization domains, polynomial rings;

Fields : field extensions, roots and factorization of polynomials, finite fields;

Vector spaces : subspaces, basis, dimension, direct sum, quotient spaces;■

Matrices : systems of linear equations, determinants, eigenvalues and eigenvectors, diagonalization, triangular forms;

Linear transformations and their representation as matrices, kernel and image, rank;

Inner product spaces, orthogonality and quadratic forms, conics and quadrics.

SAMPLE QUESTIONS FOR THE SELECTION TEST

Notation : \mathbb{R} , \mathbb{C} , \mathbb{Q} , \mathbb{Z} and \mathbb{N} denote the set of real numbers, complex numbers, rational numbers, integers and natural numbers respectively.

- (1) Let $A \subseteq \mathbb{R}^n$ and $f : A \rightarrow \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n \geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n \rightarrow \infty} f(x_n)$ exists.
- (2) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \max \{ |x|, |y| \}.$$

Show that f is a uniformly continuous function.

- (3) Let $f : [0, 1] \rightarrow [0, 1]$ be a function satisfying $|x - y| = |f(x) - f(y)|$ for all $x, y \in [0, 1]$. Show that f is surjective.
- (4) A map $f : \mathbb{R} \rightarrow \mathbb{R}$ is called open if $f(A)$ is open for every open subset A of \mathbb{R} . Show that every continuous open map of \mathbb{R} into itself is monotonic.
- (5) Let A be a bounded subset of \mathbb{R} . Suppose that for each $x \in \mathbb{R}$, there is an $\epsilon > 0$, such that $(x - \epsilon, x + \epsilon) \cap A$ is countable. Show that A is countable.
- (6) Let $S = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum |x_i|^2 = 1\}$. Let

$$A = \{(y_1, y_2, \dots, y_n) \in \mathbb{R}^n : \sum \frac{y_i}{i} = 0\}.$$

Show that the set $S + A = \{x + y : x \in S, y \in A\}$ is a closed subset of \mathbb{R}^n .

- (7) Let f be a continuous function on $[0, 1]$. Evaluate

$$\lim_{n \rightarrow \infty} \int_0^1 x^n f(x) dx$$

- (8) Let f be a differentiable function with bounded derivative on $[0, 1]$. Let P be a partition $0 = a_0 < a_1 < \dots < a_n = 1$ and $U(f, P)$ the upper Riemannian sum of f with respect to P .

Show that

$$\left| \int_0^1 f(x) dx - U(f, P) \right| \leq \sup_{0 < x < 1} |f'(x)|.$$

(9) Let $f : (0, 1) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ with } m, n \text{ relatively prime} \end{cases}$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x > \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Show that $g \circ f$ is not Riemann integrable.

(10) Let f be a continuous and nonnegative function on $[0, \infty)$. Let F be a function on $[0, \infty)$ satisfying:

- (i) $F(0) = 0$,
- (ii) $\lim_{x \rightarrow \infty} F(x) = a > 0$,
- (iii) $F'(x) = f(x)$ for every $x \in [0, \infty)$.

Show that

$$\int_0^{\infty} x f(x) dx = \int_0^{\infty} (a - F(y)) dy.$$

(You may assume that interchange of order of integration is permissible.)

(11) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with $f(0) = 0$. Define

$$f_n(x) = f(nx), \text{ for } x \in \mathbb{R} \text{ and } n = 1, 2, 3, \dots$$

Suppose that $\{f_n\}$ is equicontinuous on $[0, 1]$, that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in [0, 1]$, $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \varepsilon$ for all n . Show that $f(x) = 0$ for all $x \in [0, 1]$.

(12) Find the most general curve in \mathbb{R}^2 whose normal at each point passes through $(0, 0)$. Find the particular curve through $(2, 3)$.

- (13) Find the maximum value of the function

$$f(x, y, z) = s(s - x)(s - y)(s - z),$$

where $s > 0$ is a given constant under the condition

$$x + y + z - 2s = 0,$$

and where x, y, z are restricted by the inequalities

$$x \geq 0, y \geq 0, z \geq 0,$$

$$x + y \geq z, x + z \geq y, y + z \geq x.$$

- (14) For $x, y \in \mathbb{R}$, define

$$d(x, y) = |\tan^{-1}x - \tan^{-1}y|,$$

where for any real number t , $\tan^{-1}t$ is the unique θ , $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that $\tan\theta = t$. Show the following:

- (i) d is a metric on \mathbb{R} .
 - (ii) d is not a complete metric on \mathbb{R} .
- (15) Let (X, d) be a compact metric space and $f : X \rightarrow X$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is onto.
- (16) Let ω be an n -th root of unity such that $\omega^m \neq 1$ for any positive integer $m < n$. Show that $(1 - \omega)\dots(1 - \omega^{n-1}) = n$ [Hint : Consider the polynomial $z^n - 1$].

Hence deduce the following : if A_1, A_2, \dots, A_n are the vertices of a regular n -gon inscribed in a unit circle, prove that

$$l(A_1A_2)l(A_1A_3)\dots l(A_1A_n) = n,$$

where $l(AB)$ denotes the length of a line segment AB .

- (17) Find all homomorphisms of the group \mathbb{Q} to the group \mathbb{Z} (both are groups under addition).
- (18) Let G be a nonabelian group of order 55. How many subgroups of order 11 does it have? Using this information or otherwise compute the number of subgroups of order 5.
- (19) Let R be a commutative ring with 1. Let $R^* = \{r \in R : \text{there exists } s \in R \text{ with } rs = sr = 1\}$ denote the group of units of R .
- (a) Let $k[x]$ be the polynomial ring in one variable x over the

field k . What is $k[x]^*$?

(b) Let $A = \frac{\mathbb{Z}}{4\mathbb{Z}}$, where $\frac{\mathbb{Z}}{4\mathbb{Z}} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are the integers modulo 4. Let $A[x]$ be the polynomial ring in one variable x over the ring A . Give an example of an element of $A[x]^*$ which is not $\bar{1}$ or $\bar{3}$.

(c) Compute $A[x]^*$ with $A = \frac{\mathbb{Z}}{4\mathbb{Z}}$. (Hint: you may use the natural ring homomorphism $\frac{\mathbb{Z}}{4\mathbb{Z}}[x] \rightarrow \frac{\mathbb{Z}}{2\mathbb{Z}}[x]$, and part (a) for the field $k = \frac{\mathbb{Z}}{2\mathbb{Z}}$).

- (20) Let $n \in \mathbb{N}$ and p be a prime number. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_\ell x^\ell$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$, where $a_i, b_j \in \mathbb{Z}/p^n\mathbb{Z}$, for all $0 \leq i \leq \ell$, $0 \leq j \leq m$. Suppose that $fg = 0$. Prove that $a_ib_j = 0$ for all $0 \leq i \leq \ell$, $0 \leq j \leq m$.
- (21) Let R be the ring of all continuous real-valued functions on \mathbf{R} where the addition and the multiplication in R are defined as

$$(f + g)(x) = f(x) + g(x) , (f.g)(x) = f(x)g(x).$$

Let $f_1, \dots, f_n \in R$.

- (i) If the f_i 's have a common zero, show that the ideal generated by them is a proper ideal.
- (ii) If the f_i 's do not have a common zero, prove that the constant function 1 is in the ideal generated by them.
- (22) Let a_1, a_2, \dots, a_n be n distinct odd integers. Prove that the polynomial $f(x) = (x - a_1)(x - a_2)\dots(x - a_n) + 1$ is irreducible in $\mathbb{Z}[x]$.
- (23) Show that the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are isomorphic as vector spaces over \mathbb{Q} but not isomorphic as fields.
- (24) Suppose $f \in F[x]$ be an irreducible polynomial of degree 5, where F is a field. Let K be a quadratic field extension of F , that is, $[K : F] = 2$. Prove that f remains irreducible over K .
- (25) Let $k[x, y]$ be the polynomial ring in two variables x and y over a field k . Prove that any ideal of the form $I = (x - a, y - b)$ for $a, b \in k$ is a maximal ideal of this ring. What is the vector space dimension (over k) of the quotient space $k[x, y]/I$?

- (26) Let A be a $n \times n$ symmetric matrix of rank 1 over the complex numbers \mathbb{C} . Show that $A = \alpha \mathbf{u} \mathbf{u}^t$ for some non-zero scalar $\alpha \in \mathbb{C}$ and a non-zero vector $\mathbf{u} \in \mathbb{C}^n$ (where \mathbf{u}^t is the transpose of \mathbf{u}).
- (27) Let A be any 2×2 matrix over \mathbb{C} and let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be any polynomial over \mathbb{C} . Show that $f(A)$ is a matrix which can be written as $c_0I + c_1A$ for some $c_0, c_1 \in \mathbb{C}$, where I is the identity matrix.
- (28) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation. Show that there is a line L through origin such that $T(L) = L$.
- (29) Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0$ for all $(i, j) \neq (1, 2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.
- (30) Let $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Find all invariant subspaces of B (a subspace $\mathcal{M} \subset \mathbb{C}^3$ is said to be invariant for B if $Bx \in \mathcal{M}$ for all $x \in \mathcal{M}$).

MODEL QUESTION PAPER

Time : 2 hours

- a): Attempt any three questions from each group.
 b): Each question carries equal weightage.

GROUP A (ANY THREE)

- (1) Suppose that f is a real-valued continuous function defined on \mathbb{R} and $f(x+1) = f(x)$ for all $x \in \mathbb{R}$.
 (a) Show that f is bounded above and below and achieves its maximum and minimum.
 (b) Show that f is uniformly continuous on \mathbb{R} .
 (c) Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.
- (2) Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function for each $n \geq 1$, with $|f'_n(x)| \leq 1$ for all n and x . Assume also that $\lim_{n \rightarrow \infty} f_n(x) = g(x)$ exists for all $x \in \mathbb{R}$. Prove that $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous.
- (3) Suppose that $\{a_k\}_{k=1}^{\infty}$ is a bounded sequence of nonnegative real numbers. Show that $\frac{1}{n} \sum_{k=1}^n a_k \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\frac{1}{n} \sum_{k=1}^n a_k^2 \rightarrow 0$ as $n \rightarrow \infty$.
- (4) Consider the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0.$$

Suppose that $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions of this differential equation. Suppose also that there exist $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$ and $y_1(x_1) = 0 = y_1(x_2)$. Show that there is x_3 in \mathbb{R} such that $x_1 < x_3 < x_2$ with $y_2(x_3) = 0$.

- (5) Let (X, d) be a metric space. For a closed subset A of X , define the function d_A by

$$d_A(x) := \inf\{d(x, y) : y \in A\}.$$

Prove that

- (i) $|d_A(x_1) - d_A(x_2)| \leq d(x_1, x_2)$ for all $x_1, x_2 \in X$.
(ii) $d_A(x) = 0$ if and only if $x \in A$.

GROUP B (ANY THREE)

- (1) For a prime p , $\mathbb{F}_p (= \mathbb{Z}/p\mathbb{Z})$ denotes the field of integers modulo p . Determine all primes p for which the system of equations

$$\begin{aligned} 8x + 3y &= 10 \\ 2x + 6y &= -1 \end{aligned}$$

- (i) has no solution in \mathbb{F}_p ;
(ii) has exactly one solution in \mathbb{F}_p ;
(iii) has more than one solution in \mathbb{F}_p . In case (iii), how many solutions does the system have in \mathbb{F}_p ?

- (2) Let $a, b \in \mathbb{Z}$ and let d be the G.C.D. of a and b . Let D denote the subring of \mathbb{Q} defined by

$$D = \left\{ \frac{r}{d^k} \in \mathbb{Q} \mid r \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\} \right\}.$$

Show that $(a^m, b^n)D = D$ for all $m, n \in \mathbb{N}$; where $(a^m, b^n)D$ denotes the ideal $\{a^m\alpha + b^n\beta \mid \alpha, \beta \in D\}$.

- (3) A field L is called an algebraic extension of a subfield k if, for each $\alpha \in L$, there exists a nonzero polynomial $f(x) \in k[x]$ such that $f(\alpha) = 0$.

Suppose that k is a field and L is an algebraic extension of k . Show that any subring R of L , such that $k \subseteq R \subseteq L$, is a field.

- (4) Let A be an $n \times n$ real matrix such that $A^2 = I$, but $A \neq \pm I$ (where I denotes the $n \times n$ -identity matrix). Show that
- (i) A has two eigenvalues λ_1, λ_2 .
 - (ii) Every element $x \in \mathbb{R}^n$ can be expressed uniquely as $x_1 + x_2$, where $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$.
- (5) Let N be a normal subgroup of a finite group G such that the index $[G : N]$ is relatively prime to $|N|$, where $|N|$ denotes the order of N . Show that there is no other subgroup of G of order $|N|$.