

2005 – MA

Test Paper Code : MA

Time : 3 Hours

Max. Marks : 300

INSTRUCTIONS

- The question-cum-answer book has 44 pages and has 32 questions. Please ensure that the copy of the question-cum-answer book you have received contains all the questions.
- Write your **Roll Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
- Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on page No. 7. Do not write anything else on this page.
- Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
 - For each objective question, you will be awarded **6 (six)** marks if you have written only the correct answer.
 - In case you have not written any answer for a question you will be awarded **0 (zero)** mark for that question.
 - In all other cases, you will be awarded **-2 (minus two)** marks for the question.
- Answer the subjective question only in the space provided after each question.
- Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing later only will be evaluated.
- All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
- All rough work should be done in the space provided and scored out finally.
- No supplementary sheets will be provided to the candidates.
- Logarithmic Tables / Calculator of any kind / cellular phone / pager / electronic gadgets are not allowed.**
- The question-cum-answer book must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this book.
- Refer to special instruction/useful data on the reverse.

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READ THE INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER

Name :

Test Centre :

Do not write your Roll Number or Name anywhere else in this question-cum-answer book.

I have read all the instructions and shall abide by them.

.....
Signature of the Candidate

I have verified the information filled by the Candidate above.

.....
Signature of the Invigilator

IMPORTANT NOTE FOR CANDIDATES

Objective Part :

Attempt **ALL** the objective questions (Questions 1 – 15). Each of these questions carries six marks. Write the answers to the objective questions **ONLY** in the *Answer Table for Objective Questions* provided on page 7.

Subjective Part :

Attempt **ALL** questions in the *Core Section* (Questions 16 – 22). Questions 16 – 21 carry twenty one marks each and Question 22 carries twelve marks. There are Five Optional Sections (A, B, C, D and E). Each Optional Section has two questions, each of the questions carries eighteen marks. Attempt both questions from any two Optional Sections. Thus in the Subjective Part, attempt a total of 11 questions.

1. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $b_n = a_{2n}$ and $c_n = a_{2n+1}$. Then $\{a_n\}$ is convergent
 - (A) implies $\{b_n\}$ is convergent but $\{c_n\}$ need not be convergent
 - (B) implies $\{c_n\}$ is convergent but $\{b_n\}$ need not be convergent
 - (C) implies both $\{b_n\}$ and $\{c_n\}$ are convergent
 - (D) if both $\{b_n\}$ and $\{c_n\}$ are convergent

2. An integrating factor of $x \frac{dy}{dx} + (3x + 1)y = xe^{-2x}$ is
 - (A) xe^{3x}
 - (B) $3xe^x$
 - (C) xe^x
 - (D) x^3e^x

3. The general solution of $x^2 \frac{d^2 y}{dx^2} - 5x \frac{dy}{dx} + 9y = 0$ is

(A) $(c_1 + c_2 x)e^{3x}$

(B) $(c_1 + c_2 \ln x)x^3$

(C) $(c_1 + c_2 x)x^3$

(D) $(c_1 + c_2 \ln x)e^{x^3}$

(Here c_1 and c_2 are arbitrary constants.)

4. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. If $\phi(x, y, z)$ is a solution of the Laplace equation then the vector field $\left(\vec{\nabla}\phi + \vec{r}\right)$ is

(A) neither solenoidal nor irrotational

(B) solenoidal but not irrotational

(C) both solenoidal and irrotational

(D) irrotational but not solenoidal

5. Let $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, S be the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ be the inward unit normal vector to S . Then $\iint_S \vec{F} \cdot \hat{n} dS$ is equal to

(A) 4π

(B) -4π

(C) 8π

(D) -8π

6. Let A be a 3×3 matrix with eigenvalues 1, -1 and 3. Then
- (A) $A^2 + A$ is non-singular
 - (B) $A^2 - A$ is non-singular
 - (C) $A^2 + 3A$ is non-singular
 - (D) $A^2 - 3A$ is non-singular
7. Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be a linear transformation and I be the identity transformation of $\mathbf{R}^3$. If there is a scalar c and a non-zero vector $x \in \mathbf{R}^3$ such that $T(x) = cx$, then $\text{rank}(T - cI)$
- (A) cannot be 0
 - (B) cannot be 1
 - (C) cannot be 2
 - (D) cannot be 3
8. In the group $\{1, 2, \dots, 16\}$ under the operation of multiplication modulo 17, the order of the element 3 is
- (A) 4
 - (B) 8
 - (C) 12
 - (D) 16
9. A ring R has maximal ideals
- (A) if R is infinite
 - (B) if R is finite
 - (C) if R is finite with at least 2 elements
 - (D) only if R is finite

10. The integral $\int_0^1 \left[\int_0^{1-z} \left(\int_0^2 dx \right) dy \right] dz$ is equal to

(A) $\int_0^1 \left[\int_0^{1-y} \left(\int_0^2 dx \right) dz \right] dy$

(B) $\int_0^1 \left[\int_0^{1-y} \left(\int_0^1 dx \right) dz \right] dy$

(C) $\int_0^2 \left[\int_0^2 \left(\int_0^{1-z} dx \right) dz \right] dy$

(D) $\int_0^2 \left[\int_0^2 \left(\int_0^{1-y} dx \right) dz \right] dy$

11. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be continuous and $g, h: \mathbf{R}^2 \longrightarrow \mathbf{R}$ be differentiable. Let $F(u, v) = \int_v^u f(t) dt$,

where $u = g(x, y)$ and $v = h(x, y)$. Then $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} =$

(A) $f(g(x, y)) \left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \right] - f(h(x, y)) \left[\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \right]$

(B) $f(h(x, y)) \left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \right] - f(g(x, y)) \left[\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y} \right]$

(C) $f(h(x, y)) \left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \right] + f(g(x, y)) \left[\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y} \right]$

(D) $f(g(x, y)) \left[\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y} \right] + f(g(x, y)) \left[\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y} \right]$

12. Let $y = f(x)$ be a smooth curve such that $0 < f(x) < K$ for all $x \in [a, b]$. Let

L = length of the curve between $x = a$ and $x = b$

A = area bounded by the curve, x -axis, and the lines $x = a$ and $x = b$

S = area of the surface generated by revolving the curve about x -axis between $x = a$ and $x = b$

Then

(A) $2\pi KL < S < 2\pi A$

(B) $S \leq 2\pi A < 2\pi KL$

(C) $2\pi A \leq S < 2\pi KL$

(D) $2\pi A < 2\pi KL < S$

13. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $f(t) = t^2$ and let U be any non-empty open subset of $\mathbf{R}$.

Then

(A) $f(U)$ is open

(B) $f^{-1}(U)$ is open

(C) $f(U)$ is closed

(D) $f^{-1}(U)$ is closed

14. Let $f : (-1, 1) \longrightarrow \mathbf{R}$ be such that $f^{(n)}(x)$ exists and $|f^{(n)}(x)| \leq 1$ for every $n \geq 1$ and for every $x \in (-1, 1)$. Then f has a convergent power series expansion in a neighbourhood of

- (A) every $x \in (-1, 1)$
- (B) every $x \in \left(-\frac{1}{2}, 0\right)$ only
- (C) no $x \in (-1, 1)$
- (D) every $x \in \left(0, \frac{1}{2}\right)$ only

15. Let $a > 1$ and $f, g, h : [-a, a] \longrightarrow \mathbf{R}$ be twice differentiable functions such that for some c with $0 < c < 1 < a$,

$$f(x) = 0 \text{ only for } x = -a, 0, a;$$

$$f'(x) = 0 = g(x) \text{ only } x = -1, 0, 1;$$

$$g'(x) = 0 = h(x) \text{ only for } x = -c, c.$$

The possible relations between f, g, h are

- (A) $f = g'$ and $h = f'$
- (B) $f' = g$ and $g' = h$
- (C) $f = -g'$ and $h' = g$
- (D) $f = -g'$ and $h' = f$

CORE SECTION

16. (a) Solve the initial value problem

$$\frac{d^2 y}{dx^2} - y = x(\sin x + e^x), \quad y(0) = y'(0) = 1 \quad (12)$$

- (b) Solve the differential equation

$$(2y \sin x + 3y^4 \sin x \cos x)dx - (4y^3 \cos^2 x + \cos x)dy = 0 \quad (9)$$



17. Let G be a finite abelian group of order n with identity e . If for all $a \in G$, $a^3 = e$, then, by induction on n , show that $n = 3^k$ for some nonnegative integer k . (21)

18. (a) Let $f : [a, b] \longrightarrow \mathbf{R}$ be a differentiable function. Show that there exist points $c_1, c_2 \in (a, b)$ such that

$$2f(c_1)f'(c_1) = f'(c_2)[f(a) + f(b)] \quad (9)$$

- (b) Let

$$f(x, y) = \begin{cases} (x^2 + y^2) [\ln(x^2 + y^2) + 1] & \text{for } (x, y) \neq (0, 0) \\ \alpha & \text{for } (x, y) = (0, 0) \end{cases}$$

Find a suitable value for α such that f is continuous. For this value of α , is f differentiable at $(0, 0)$? Justify your claim. (12)

19. (a) Let S be the surface $x^2 + y^2 + z^2 = 1, z \geq 0$. Use Stoke's theorem to evaluate

$$\int_C [(2x - y)dx - y dy - z dz]$$

where C is the circle $x^2 + y^2 = 1, z = 0$, oriented anticlockwise. (12)

- (b) Show that the vector field $\vec{F} = (2xy - y^4 + 3)\hat{i} + (x^2 - 4xy^3)\hat{j}$ is conservative. Find its potential and also the work done in moving a particle from $(1,0)$ to $(2,1)$ along some curve. (9)



20. Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be defined by $T(x, y, z) = (y + z, z, 0)$. Show that T is a linear transformation. If $v \in \mathbf{R}^3$ is such that $T^2(v) \neq 0$, then show that $B = \{v, T(v), T^2(v)\}$ forms a basis of $\mathbf{R}^3$. Compute the matrix of T with respect to B . Also find a $v \in \mathbf{R}^3$ such that $T^2(v) \neq 0$. (21)

21. (a) For each $n \in \mathbf{N}$, define $f_n : [-1, 1] \longrightarrow \mathbf{R}$ by

$$f_n(x) = \begin{cases} 4n^2x & \text{for } x \in \left[0, \frac{1}{2n}\right) \\ -4n^2\left(x - \frac{1}{n}\right) & \text{for } x \in \left[\frac{1}{2n}, \frac{1}{n}\right) \\ 0 & \text{for } x \in \left[\frac{1}{n}, 1\right] \end{cases}$$

Compute $\int_0^1 f_n(x) dx$ for each n . Analyse pointwise and uniform convergence of the sequence of functions $\{f_n\}$. (12)

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function with $|f(x) - f(y)| \geq |x - y|$ for every $x, y \in \mathbf{R}$. Is f one-one? Show that there cannot exist three points $a, b, c \in \mathbf{R}$ with $a < b < c$ such that $f(a) < f(c) < f(b)$. (9)

22. Find the volume of the cylinder with base as the disk of unit radius in the xy -plane centred at $(1, 1, 0)$ and the top being the surface $z = \left[(x-1)^2 + (y-1)^2 \right]^{3/2}$. (12)

OPTIONAL SECTION : A

23. (a) Bag A contains 3 white and 4 red balls, and bag B contains 6 white and 3 red balls. A biased coin, twice as likely to come up heads as tails, is tossed once. If it shows head, a ball is drawn from bag A, otherwise, from bag B. Given that a white ball was drawn, what is the probability that the coin came up tail? (9)
- (b) Let the random variables X and Y have the joint probability density function $f(x, y)$ given by

$$f(x, y) = \begin{cases} y^2 e^{-y(x+1)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Are the random variables X and Y independent? Justify your answer. (9)

24. (a) Let $X_1, X_2, \dots, X_n$ be independently identically distributed random variables (rv's) with common probability density function (pdf) $f_X(x, \theta) = \frac{1}{\theta} e^{-x/\theta}; x > 0, \theta > 0$. Obtain the moment generating function (mgf) of $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$. Also find the mgf of the rv $Y = 2n\bar{X}/\theta$. (9)
- (b) Let $X_1, X_2, \dots, X_9$ be an independent random sample from $N(2, 4)$ and Y_1, Y_2, Y_3, Y_4 be an independent random sample from $N(1, 1)$. Find $P(\bar{X} > \bar{Y})$, where $\bar{X}$ and $\bar{Y}$ are sample means. (9)
- [Given $P(Z > 1.2) = 0.1151$, where $Z \sim N(0, 1)$]

OPTIONAL SECTION : B

25. (a) Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution having pdf

$$f(x; x_0, \alpha) = \begin{cases} \frac{\alpha x_0^\alpha}{x^{\alpha+1}} & \text{for } x > x_0 \\ 0 & \text{otherwise} \end{cases}$$

where $x_0 > 0, \alpha > 0$. Find the maximum likelihood estimator of α if x_0 is known. (9)

- (b) Let $X_1, X_2, \dots, X_5$ be a random sample from the standard normal population. Determine the constant c such that the random variable

$$Y = \frac{c(X_1 + X_2)}{\sqrt{X_3^2 + X_4^2 + X_5^2}}$$

will have a t -distribution.

(9)

26. (a) A random sample of size $n = 1$ is drawn from pdf $f_X(x, \theta) = \frac{1}{\theta} e^{-x/\theta}$; $x > 0$, $\theta > 0$. It is decided to test $H_0 : \theta = 5$ against $H_1 : \theta = 7$ based on the criterion: reject H_0 if the observed value is greater than 10. Obtain the probabilities of type I and type II errors. (9)
- (b) Let $X_1, X_2, \dots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$. Find the best test for testing $H_0 : \mu = 0, \sigma^2 = 1$ against $H_1 : \mu = 1, \sigma^2 = 4$. (9)

OPTIONAL SECTION : C

27. (a) Let $f, g : \mathbf{R} \longrightarrow \mathbf{R}$ be such that for $x, y \in \mathbf{R}$,

$$\phi(x + iy) = e^x [f(y) + ig(y)]$$

is an analytic function. Find a differential equation of order 2 satisfied by f . (9)

(b) Compute $\int_{|z+1|=2} (2z+1)e^{(\sqrt{2} + 1/z)} dz$. (9)

28. (a) Let $f(z)$ be analytic in the whole complex plane such that for all $r > 0$,

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq \sqrt{r}$$

Find $\frac{f^{(n)}(0)}{n!}$ for all $n \geq 0$. (9)

- (b) Find all values of $\alpha \in \mathbf{C}$ such that $f(z) = (z + \bar{z})^2 + 2\alpha|z|^2 + \alpha(\bar{z})^2$ is analytic at some point z having non-zero real part. (9)

OPTIONAL SECTION : D

29. A hemispherical bowl of radius 12 cm is fixed such that its rim is horizontal. A light rod of length 20 cm with weights w and W attached to its two ends is placed inside the bowl. In equilibrium, the weight w is just touching the rim of the bowl. Find the ratio $w : W$. (18)



30. A uniform ladder of length $2a$ and mass m lies in a vertical plane with one end against a smooth wall, the other end being supported on a horizontal floor. The ladder is released from rest when inclined at an angle α to the horizontal. Find the inclination of the ladder to the horizontal when it ceases to touch the wall. (18)

OPTIONAL SECTION : E

31. (a) Estimate the error in evaluating the integral $\int_0^8 (1+x^2)e^{-x} dx$ by Simpson's $\frac{1}{3}$ rd rule with spacing $h = 0.25$. (9)

(b) Using Newton-Raphson method, compute the point of intersection of the curves $y = x^3$ and $y = 8x + 4$ near the point $x = 3$, correct up to 2 decimal places.

[Round-off the first iteration up to 2 decimal places for further computation] (9)

32. The polynomial $p_3(x) = x^3 + x^2 - 2$ interpolates the function $f(x)$ at the points $x = -1, 0, 1$ and 2 . If the data $f(3) = -14$ is added, find the new interpolating polynomial by using Newton's forward difference formula. Also find $f(2.5)$ by using Newton's backward difference formula with pivot value 3 . Justify whether the value obtained will be the same if pivot value 2 is taken. (18)

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