Syllabus for the M. Math. Selection Test
TEST CODE MM

Open sets, closed sets and compact sets in $\mathbb{R}$ and $\mathbb{R}^n$; convergence and divergence of sequence and series; continuity, uniform continuity, differentiability, Mean Value Theorem; pointwise and uniform convergence of sequences and series of functions, Taylor expansion, power series; integral calculus of one variable : Riemann integration, Fundamental theorem of calculus, change of variable; directional and total derivatives, Jacobians, chain rule; maxima and minima of functions of one and two variables; elementary topological notions for metric space : compactness, connectedness, completeness; elements of ordinary differential equations.

Equivalence relations and partitions; primes and divisibility; groups: subgroups, products, quotients, homomorphisms, Lagrange's theorem, Sylow's theorems; commutative rings and fields: ideals, maximal ideals, prime ideals, quotients, congruence arithmetic, integral domains and fields of quotients, principal ideal domains, unique factorization domains, polynomial rings; field extensions, normal extensions, roots and factorization of polynomials, finite fields; vector spaces: subspaces, basis, dimension, direct sum, quotient spaces; matrices, systems of linear equations, determinants, eigenvalues and eigen vectors; diagonalization, triangular forms; linear transformations and their representation as matrices, kernel and image, rank; inner product spaces, orthogonality and quadratic forms, conics and quadrics.

Sample Questions for the Selection Test

Notation : $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers, integers and natural numbers respectively.

(1) Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n\geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n\to\infty} f(x_n)$ exists.

(2) Let $N > 0$ and let $f : [0,1] \to [0,1]$ be denoted by $f(x) = 1$ if $x = 1/i$ for some integer $i \leq N$ and $f(x) = 0$ for all other values of x. Show that f is Riemann integrable.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \max\{|\, x\,|, |\, y\,|\}\,.$$

Show that f is a uniformly continuous function.

(4) Let $A \subseteq \mathbb{R}^n$ be a closed and bounded set. Let $f : A \to A$ be such that $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\|$, for all $\boldsymbol{x}, \boldsymbol{y} \in A$, where $\|\boldsymbol{x}\|^2 = \sum_{i=1}^n |x_i|^2$ for $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Show that f is onto.

(5) Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational ,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ with } m, n \text{ relatively prime} \end{cases}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x > \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Show that $g \circ f$ is not Riemann integrable.

(6) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$. Define

$$f_n(x) = f(nx), \text{ for } x \in \mathbb{R} \text{ and } n = 1, 2, 3, \ldots$$

Suppose that $\{f_n\}$ is equicontinuous on $[0,1]$, that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in [0,1]$,

$|x-y|<\delta$, we have $|f_n(x)-f_n(y)|<\varepsilon$ for all n. Show that $f(x)=0$ for all $x\in[0,1]$.

(7) Find the most general curve in $\mathbb{R}^2$ whose normal at each point passes through $(0,0)$. Find the particular curve through $(2,3)$.

(8) Let A be a $n\times n$ symmetric matrix of rank 1 over the complex numbers $\mathbb{C}$. Show that $A=\alpha\,\boldsymbol{u}\,\boldsymbol{u}^t$ for some non-zero scalar $\alpha\in\mathbb{C}$ and a non-zero vector $\boldsymbol{u}\in\mathbb{C}^n$ (where $\boldsymbol{u}^t$ is the transpose of $\boldsymbol{u}$).

(9) Let A be any 2×2 matrix over $\mathbb{C}$ and let $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ be any polynomial over $\mathbb{C}$. Show that $f(A)$ is a matrix which can be written as c_0I+c_1A for some $c_0,c_1\in\mathbb{C}$, where I is the identity matrix.

(10) Let G be a nonabelian group of order 55. How many subgroups of order 11 does it have? Using this information or otherwise compute the number of subgroups of order 5.

(11) Let $n\in\mathbb{N}$ and p be a prime number. Let $f(x)=a_0+a_1x+a_2x^2+\cdots+a_\ell x^\ell$ and $g(x)=b_0+b_1x+b_2x^2+\cdots+b_mx^m$, where $a_i,b_j\in\mathbb{Z}/p^n\mathbb{Z}$, for all $0\le i\le\ell$, $0\le j\le m$. Suppose that $fg=0$. Prove that $a_ib_j=0$ for all $0\le i\le\ell$, $0\le j\le m$.

(12) Suppose $f\in F[x]$ be an irreducible polynomial of degree 5, where F is a field. Let K be a quadratic field extension of F, that is, $[K:F]=2$. Prove that f remains irreducible over K.

(13) Let $k[x,y]$ be the polynomial ring in two variables x and y over a field k. Prove that any ideal of the form $I=(x-a,y-b)$ for $a,b\in k$ is a maximal ideal of this ring. What is the vector space dimension (over k) of the quotient space $k[x,y]/I$?

(14) Consider the two fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$, where $\mathbb{Q}$ is the field of rational numbers. Show that they are isomorphic as vector spaces but not isomorphic as fields.

(15) Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be a linear transformation. Show that there is a line L such that $T(L)=L$

(16) Let $A=(a_{ij})$ be a $n\times n$ matrix such that $a_{ij}=0$ for $i\ge j$. Show that $A^n=0$.

(17) Let $a_1, a_2, ..., a_n$ be n distinct integers. Prove that the polynomial $f(x) = (x - a_1)(x - a_2)...(x - a_n) + 1$ is irreducible in $\mathbb{Z}[x]$.

(18) Let ω be an n-th root of unity such that $\omega^m \neq 1$ for any positive integer $m < n$. Show that $(1 - \omega)...(1 - \omega^{n-1}) = n$ [Hint : Consider the polynomial $z^n - 1$].
Hence deduce the following : if $A_1, A_2, ..., A_n$ are the vertices of a regular n-gon inscribed in a unit circle, prove that

$$l(A_1A_2)l(A_1A_3)...l(A_1A_n) = n,$$

where $l(AB)$ denotes the length of a line segment AB.

(19) Let $f(x)$ be a non-constant polynomial with integer coefficients. Show that the set $S = \{f(n)|n \in \mathbb{N}\}$ has infinitely many composite numbers.

(20) Determine the integers n for which there exist $x, y \in \mathbb{Z}/n\mathbb{Z}$ satisfying the pair of equations $x + y = 2, 2x - 3y = 3$.