Test Code MS (Short answer type) 2008

Syllabus for Mathematics

Permutations and Combinations. Binomial and multinomial theorem. Theory of equations. Inequalities.

Determinants, matrices, solution of linear equations and vector spaces.

Trigonometry, Coordinate geometry of two and three dimensions.

Geometry of complex numbers and De Moivre's theorem. Elements of set theory.

Convergence of sequences and series. Power series. Functions, limits and continuity of functions of one or more variables.

Differentiation, Leibnitz formula, maxima and minima, Taylor's theorem. Differentiation of functions of several variables. Applications of differential calculus.

Indefinite integral, Fundamental theorem of calculus, Riemann integration and properties. Improper integrals. Differentiation under the integral sign. Double and multiple integrals and applications.

Syllabus for Statistics

Probability and Sampling Distributions

Notions of sample space and probability, combinatorial probability, conditional probability and independence, random variable and expectations, moments, standard discrete and continuous distributions, sampling distributions of statistics based on normal samples, central limit theorem, approximation of binomial to normal or Poisson law. Bivariate normal and multivariate normal distributions.

Descriptive Statistics

Descriptive statistical measures, graduation of frequency curves, product-moment, partial and multiple correlation, Regression (bivariate and multivariate).

Inference

Elementary theory and methods of estimation (unbiasedness, minimum variance, sufficiency, maximum likelihood method, method of moments). Tests of hypotheses (basic concepts and simple applications of Neyman-Pearson Lemma). Confidence intervals. Tests of regression. Elements of non-parametric inference.

Design of Experiments and Sample Surveys

Basic designs (CRD/RBD/LSD) and their analyses. Elements of factorial designs. Conventional sampling techniques (SRSWR/SRSWOR) including stratification; ratio and regression methods of estimation.

Sample Questions:

1. Let

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

   Which of the following statements are false. In each case, justify your answer.

   (a) A has only one real eigenvalue.

   (b) $\text{Rank}(A) = \text{Trace}(A)$.

   (c) Determinant of A equals the determinant of A^n for each integer $n > 1$.

2. For $k \geq 1$, let

$$a_k = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{kn} \exp\left(-\frac{1}{2}\frac{m^2}{n^2}\right).$$

   Find $\lim_{k\to\infty} a_k$.

3. Let g be a continuous function with $g(1) = 1$ such that

$$g(x + y) = 5g(x)g(y)$$

   for all x, y. Find $g(x)$.

   [**Hint:** You may use the following result.

   If f is a continuous function that satisfies $f(x + y) = f(x) + f(y)$ for all x, y, then $f(x) = xf(1)$.]

4. The unit interval $(0, 1)$ is divided into two sub-intervals by picking a point at random from inside the interval. Denoting by Y and Z the lengths of the longer and the shorter sub-intervals respectively, show that Y/Z does not have finite expectation.

5. Consider the i.i.d. sequence

$$X_1, X_2, X_3, X_4, X_5, X_6$$

where each X_i is one of the four symbols $\{a, t, g, c\}$. Further suppose that

$$P(X_1 = a) = 0.1 \qquad P(X_1 = t) = 0.2$$
$$P(X_1 = g) = 0.3 \qquad P(X_1 = c) = 0.4.$$

Let Z denote the random variable that counts the number of times that the *subsequence* **cat** occurs (*i.e.* the letters c, a and t occur consecutively and in the correct order) in the above sequence. Find $E(Z)$.

6. Let F and G be (one dimensional) distribution functions. Decide which of the following are distribution functions.

   (a) F^2,

   (b) H where $H(t) = \max\{F(t), G(t)\}$.

   Justify your answer.

7. Let X and Y be exponential random variables with parameters 1 and 2 respectively. Another random variable Z is defined as follows.

   A coin, with probability p of Heads (and probability $1 - p$ of Tails) is tossed. Define Z by

$$Z = \begin{cases} X & \text{if the coin turns Heads} \\ Y & \text{if the coin turns Tails} \end{cases}$$

   Find $P(1 \leq Z \leq 2)$.

8. Let $\underline{Y} = (Y_1, Y_2)'$ have the bivariate normal distribution $N_2(\underline{0}, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Obtain the mean and variance of $U = \underline{Y}'\Sigma^{-1}\underline{Y} - Y_1^2/\sigma_1^2$.

9. Let $X_1, \dots, X_m$ be a random sample from a uniform distribution on $\{1, 2, \dots, N\}$ where N is an unknown positive integer. Find the MLE $\widehat{N}$ of N and find its distribution function.

10. Consider a population with three kinds of individuals labelled 1, 2 and 3. Suppose the proportion of individuals of the three types are given by $f(k, \theta), k = 1, 2, 3$ where $0 < \theta < 1$ and

$$f(k,\theta) = \begin{cases} \theta^2 & \text{if } k = 1 \\ 2\theta(1-\theta) & \text{if } k = 2 \\ (1-\theta)^2 & \text{if } k = 3. \end{cases}$$

Let $X_1, X_2, \dots, X_n$ be a random sample from this population. Find the most powerful test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ $(\theta_0 < \theta_1 < 1)$.

11. Let r be the number of successes in n Bernoulli trials with unknown probability p of success. Obtain the minimum variance unbiased estimator of $p - p^2$.

12. Consider a randomized block experiment with 4 treatments and 3 replicates (blocks) and let τ_i be the effect of the ith treatment $(1 \le i \le 4)$. Find all possible covariances between the least squares estimators of the following treatment contrasts:

    (a) $\tau_1 - \tau_2$,

    (b) $\tau_1 + \tau_2 - 2\tau_3$,

    (c) $\tau_1 + \tau_2 + \tau_3 - 3\tau_4$.

* For more smaple questions, visit http://www.isical.ac.in/~deanweb/MSTATSQ.html