Test Code: RE I/RE II (Short Answer type) 2008

(Junior Research Fellowship in Economics)

The candidates for Junior Research Fellowship in Economics are required to take two short answer type tests-RE I (Mathematics) in the forenoon session and RE II (Economics) in the afternoon session.

Syllabus for RE I

1. Permutations and combinations.

2. Theory of quadratic equations.

3. Elementary set theory; Functions and relations; Matrices.

4. Convergence of sequence and series.

5. Functions of one and several variables : limits, continuity, differentiation, applications, integration of elementary functions and definite integrals.

6. Constrained and unconstrained optimization, convexity of sets and concavity and convexity of functions.

7. Elements of probability theory, discrete and continuous random variables, expectation and variance, joint, conditional and marginal distributions, distribution function of a random variable.

Syllabus for RE II

1. Theory of consumer behaviour; Theory of production; Market structure; General equilibrium and welfare economics.

2. Macroeconomic theories of income determination; Rational expectations; Phillips curve; Neo-classical growth model; Theories of business cycle.

3. Applications of microeconomics and macroeconomics to the problems of development economics, international trade and public economics.

4. Game Theory : Normal and extensive forms, Nash and sub-game perfect equilibrium.

5. Econometric theory and applications : regression analysis (including heteroscedasticity, autocorrelation and multicollinearity), least squares and maximum likelihood methods of estimation, specification bias, endogeneity and exogeneity, simultaneous system of equations, instrumental variables.

6. Poverty and inequality.

7. Elementary time series analysis.

Sample questions for RE I

1. (a) Find, with a clear explanation, the points of *local maximum* and *minimum*, if any, for the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^3 - 6x^2 + 24x.$$

   (b) Suppose $g : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$g(x) = -5 - 5x + 8x^2 - x^3.$$

   Find, with a clear explanation, the point of *global maximum* of g.

2. Prove or disprove:

   (a) The union of convex sets is convex.

   (b) The intersection of convex sets is convex.

   (c) Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. The set

$$\begin{aligned} F = \quad & \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij} x_j = b_i \ \forall i \in \{1, \ldots, m\}, \\ & x_j \geq 0 \ \forall \ j \in \{1, \ldots, n\}\} \end{aligned}$$

   is convex.

3. Consider the following definition of a convex function:

   Let A be a convex set in $\mathbb{R}$. Then $f : A \to \mathbb{R}$ is a *convex function* (on A) if for all $x^1, x^2 \in A$, and for all $0 \leq \theta \leq 1$,

   $$f\left(\theta x^1 + (1-\theta)\, x^2\right) \leq \theta f\left(x^1\right) + (1-\theta)\, f\left(x^2\right).$$

   Use this definition to prove that

   $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

   when f is a *convex* function on $\mathbb{R}$ and x_1, x_2 and x_3 are points of $\mathbb{R}$ which satisfy $x_1 < x_2 < x_3$.

4. Prove that the equation

   $$17x^7 - 19x^5 - 1 = 0$$

   has a solution x^* which satisfies $-1 < x^* < 0$.

5. A set of vectors x^1, x^2, ..., x^m (here $x^i = \left(x_1^i, x_2^i, ..., x_n^i\right) \in \mathbb{R}^n$ is a vector) is *linearly dependent* if there exist numbers $\lambda_1, \lambda_2, ..., \lambda_m$, *not all zero*, such that

   $$\lambda_1 x^1 + \lambda_2 x^2 + ... + \lambda_m x^m = 0 \text{ (the zero vector in } \mathbb{R}^n).$$

   A set of vectors is called *linearly independent* if the vectors are *not* linearly dependent.

   (a) Are the following three vectors in $\mathbb{R}^3$, $x^1 = (1, 0, 0)$, $x^2 = (1, 1, 0)$ and $x^3 = (1, 1, 1)$ linearly dependent? Explain clearly.

   (b) Let x, y and z be 3 linearly independent vectors in $\mathbb{R}^n$. Are the vectors $(x + y)$, $(y + z)$, $(z + x)$ linearly independent? Explain clearly.

6. Recall that a *sequence of real numbers*, $\{x_1, x_2, x_3, ...\}$, is an assignment of a real number to each natural number. A real number x is called the *limit* of the sequence $\{x_1, x_2, x_3, ...\}$ if given any real number $\epsilon > 0$, there is a positive integer N such that $|x_n - x| < \epsilon$ whenever $n \geq N$.

Prove or disprove the following:

(a) Let $\{x_1, x_2, x_3, ...\}$ be a convergent sequence with limit x and b be a real number. If $x_n \leq b$ for all n, then $x \leq b$.

(b) Let $\{x_1, x_2, x_3, ...\}$ be a convergent sequence with limit x and b be a real number. If $x_n < b$ for all n, then $x < b$. [Note that the inequalities are strict for this part.]

7. (a) Explain the relationship between joint, conditional and marginal distributions using Bayes' rule.

   (b) Illustrate the differences between probability mass function (p.m.f) and probability density function (p.d.f) using an example of a finite possibilities p.m.f and an infinite possibilities p.m.f.

8. On a Sunday, a resident of Dehradun decided to measure rainfall of the city in the following 6 days of the week. He found it consecutively to be 5, 6, 5, 5, 6, 6 millimeters.

   (a) What is the average rainfall?

   (b) Use this as an estimate of the average rainfall that month.

   (c) Now show the computation of its variance without computing the deviation of each observation from its mean.

9. (a) The probability that a person who sits for the National Education Test can teach well is 0.1. Those who can teach well pass the National Education Test with probability 0.9 while those who cannot teach well pass it with probability 0.6. What is the probability that a person who has passed the test can teach well?

   (b) Suppose n numbers are drawn uniformly from $[0, a]$, $(a > 0)$. Show that the expected value of the second highest number is $a \cdot \left(\frac{n-1}{n+1}\right)$.

Sample questions for RE II

1. Suppose the world consists of two countries: India and Australia. Each country is endowed with two goods: sari and cricket bat. The (sari, cricket bat) endowments of India and Australia are respectively (400, 180) and (100, 60). The consumers in country i have the utility function: $U = \sqrt{S_i.C_i}$, where S_i and C_i denote the consumption of sari and cricket bat respectively by a country i consumer.

   (a) Initially the two countries do not trade. In an autarky equilibrium, how much of each good these countries consume and what will be the prevailing relative prices?

   (b) Suppose the two countries open up free trade between them. What will be the equilibrium terms of trade? Which country will export which good and why? Also, determine the volumes exported and imported.

   (c) Discuss qualitatively if both countries stand to gain from free trade, and the sources of their welfare gains or losses.

2. A radio station runs the following contest. Every participant has to send an integer between 0 and 100 (both inclusive). A **target integer** is defined by multiplying the highest received integer by $\frac{9}{10}$, and then taking only the integer part of the number. As an example, if the highest received entry is 81, then the target integer is 72. All participants who sent the target integer equally divide a prize of Rs. 1000. Others get nothing. Suppose 100 participants took part in this game.

   Modeling this as a simultaneous move game and restricting attention to pure strategies only, answer the following questions with clear explanations for your answers.

   (a) Identify a Nash equilibrium of this game. What are the payoffs of the participants in this equilibrium?

   (b) Is this Nash equilibrium unique?

3. The city of Dilly-dally has a population of one lakh identical individuals all of whom like to drive around in cars. An individual's utility is

$$U = m + 200d - d^2h^3,$$

where m is the individual's endowment of rupees, d is the rupees spent by an individual on driving, and h is the average expenditure on driving (averaged over the entire population).

(a) Find the Nash equilibrium level of driving d_N chosen by each person and the associated level of utility U_N (in terms of m). [A fact you will find useful is that $\sqrt{10} = 3.16$ (approximately)].

(b) Since the inhabitants of Dilly-dally complained about the congestion, the city government proposed constructing a flyover at a cost of c lakhs which would reduce congestion, thus changing the utility function of a consumer to

$$U = m - c + 200d - \frac{d^2h^3}{2}.$$

Let the new equilibrium level of driving be d_F. Is $d_F > d_N$? Assuming that the construction cost of c lakhs is financed by equal lump-sum taxes on the one lakh users of the flyover, below what level of c is the associated new level of utility $U_F > U_N$? . (Useful approximation: $200^{\frac{1}{4}} = 3.76$.)

(c) If, instead of a flyover, a unit tax on driving (which is returned to the inhabitants in a lump-sum manner), were used, what level of the tax would achieve the Pareto-optimum? Is this better than the flyover proposal? (Useful approximation: $40^{\frac{1}{4}} = 2.5$.).

4. (a) Consider an initial Cournot equilibrium of 2 firms; market demand (for a homogeneous product) is assumed to be downward sloping and firm i has constant unit cost of production c_i, $c_1 \geq c_2 > 0$. Now suppose that firm 1's output is taxed and firm 2's output is subsidized, both at the same rate, t, per unit.

What will be its implication for the net tax collection of the government?

(b) Consider a duopoly market for homogeneous products. The market demand for the product is given by $p = 100 - q$, and the unit cost of production is zero. Given the above demand function, assume that the firms have already produced their output levels, fixed at 100/3 and 100/3. They will now decide their prices simultaneously. The demand facing firm i, in case it charges a higher price, p_i, is $\max\{0, 100 - p_i - q_j\}$, where q_j denotes the output supplied by firm j. What prices will they quote? Interpret this result.

5. A monopolist sells a book to heterogenous consumers with consumer i having an income of M_i, where M_i is uniformly distributed over $[0, \overline{M}]$. Each consumer buys at most one book. In case he buys the book, the utility function of the i-th consumer is $(M_i - p_n)\delta_n$, where p_n is the price of the book and $\delta_n > 0$. If he does not buy the book his utility is zero. The monopolist has constant marginal cost c and chooses p_n to maximize profits.

   (a) Derive the market demand function faced by the monopolist, and find out his optimal price.

   (b) Now suppose that the book may be freely photocopied by the consumers at a constant per unit cost $q < c$. Accordingly, a consumer has three choices: (i) buy a new book, (ii) buy a photocopy, or (iii) not buy the book at all. In case (ii), the consumer has a utility of $(M_i - q)\delta_o$, where $\delta_n > \delta_o > 0$.

   Set up the monopolist's optimization problem and find out the optimal price.

6. Consider a team production problem with $n \geq 2$ agents. Each agent i can put in an effort $e_i \in [0, E]$, where E is finite and $E > 1$. The cost to i of putting in effort e_i is $\frac{\alpha_i . e_i^2}{2}$, where $\alpha_i = [\beta + \sum_{j \neq i} e_j]^{-1}$ and $\beta \in [0, 1]$. When each agent i puts in effort e_i, the total output from the team project is $\sum_{i=1}^{n} e_i$. Suppose total output is shared equally by all agents. Suppose all agents behave as individual payoff maximisers and let e^* be the symmetric equilibrium effort level.

(a) Compute e^*.

(b) Also compute the payoff π^* received by any agent at the symmetric equilibrium.

(c) How do e^* and π^* vary as n increases? What happens to them as n increases without bound and approaches ∞?

(d) Suppose now there is a planner who can choose an arbitrary effort level $e \in [0, E]$, and ensure that all agents put in this effort level. The planner chooses e to maximise the sum of all agents' net payoffs (total output minus total costs). Let the planner's optimal choice be denoted by $\overline{e}$.

Compute $\overline{e}$.

7. Consider the following extensive form production function which has a constant elasticity of substitution (CES) between labor and capital

$$Y = F(K, L) = A\left\{a(bK)^{\theta} + (1-a)[(1-b)L]^{\theta}\right\}^{\frac{1}{\theta}},$$

where $0 < a < 1$, $0 < b < 1$, and $\theta < 1$.

(a) Show that the above production function exhibits constant returns to scale for all values of θ.

(b) Derive an expression for the elasticity of substitution between K and L.

(c) Given the above production function, derive the intensive form production function, i.e., find $y = f(k)$, where $y = \frac{Y}{L}$ and $k = \frac{K}{L}$.

(d) Using the intensive form production function, compute expressions for

(i) the marginal product of capital, $f'(k)$,

(ii) the average product of capital, $\frac{f(k)}{k}$.

(e) Consider the case when $0 < \theta < 1$. What happens to the marginal and average products of capital as $k \to \infty$? Is there endogenous steady state growth ? Explain your answer.

(f) Now consider the case when $\theta < 0$. What happens to the marginal and average products of capital as $k \to \infty$? Is there endogenous steady state growth ? Explain your answer.

(g) Does the CES model exhibit the conditional convergence property? Explain your answer with the help of a diagram.

8. (a) "At given interest rate, a given increase in real income would lead to a less than proportionate increase in the real demand for money." Which theory of demand for money would imply this and how?

   (b) What is the Lucas supply function and what does it say intuitively? Lay down any (one) micro foundation of this function.

9. A student wishes to estimate the average daily rainfall in the month of July in Jaipur. The student assumes that rainfall on any particular July day (in Jaipur) is randomly distributed with mean μ and variance σ^2. The student randomly picks 6 days in this month and obtains the following measurements (all in millimetres):

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
5	6	5	5	6	6

   (a) What would you suggest as estimators of μ and σ^2? Why?

   (b) Using these estimators, obtain estimates of μ and σ^2.

   (c) If g is an estimator of μ, then the mean square error of g is defined as $E(g-\mu)^2$. What is the mean square error for the estimator of μ that you proposed?

   (d) Can there be an estimator with a lower mean square error than the one you proposed?

   (e) Suggest a procedure for testing the hypothesis that average daily rainfall in Jaipur is 8 mm. If this procedure requires additional assumptions, specify them and argue how they justify the procedure.

10. (a) Let the time series y_t be defined by

$$y_t = \beta_0 + \beta_1 t + X_t, \ t = 1, 2, \cdots$$

where $X_t = a_t + 0.6a_{t-1}$ and a_t $(t = 0, 1, 2, \cdots)$ is a sequence of normal independent $(0, \sigma^2)$ random variables and β_0 and β_1 are fixed (unknown) parameters.

Find the mean, variance and auto-covariance function of y_t.

(b) Show that the AR (3) process

$$X_t = X_{t-1} - cX_{t-2} + cX_{t-3} + Z_t, \ Z_t \sim \text{white noise } (0, \sigma^2)$$

is nonstationary for all values of c.

11. (a) Consider a standard linear regression model of y_t on a constant and a regressor where the dependent variable, y_t, has a sample mean 19.32. Suppose that a new dependent variable is defined as $y_t^* = y_t + 15$, and the same linear regression is done using y_t^* instead of y_t as the regressand.

Find how the estimates of the constant term and slope parameter as well as the value of R^2 would change in the latter regression, as compared to the former?

(b) Consider the following two regressions:

$$Y = X\beta + \epsilon$$

and

$$Y = X\beta + Zv + u$$

where y is $n \times 1$, X is $n \times k_1$, Z is $n \times k_2$, ϵ is $n \times 1$ and u is $n \times 1$, n being the number of observations. ϵ and u are assumed to follow the standard assumptions of the classical linear regression model.

(i) Find the ordinary least squares estimators of β from the two regressions and check if these two estimators are the same.

(ii) In case, you think that these are not the same, then find a condition involving X and Z for which the estimators of

β are identical. [You may use the following result involving matrices:

$$\begin{pmatrix} A & B \\ \\ B' & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + FE^{-1}F' & -FE^{-1} \\ \\ -E^{-1}F' & E^{-1} \end{pmatrix}$$

where $E = D - B'A^{-1}B$, $F = A^{-1}B$ and A and D are symmetric matrices.]