INMO 2004 - Solutions

1. Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively. Suppose

   (a) BD bisects KM at Q;

   (b) $QA = QB = QC = QD$; and

   (c) $LK/LM = CD/CB$.

   Prove that $ABCD$ is a **square**.

   Solution:

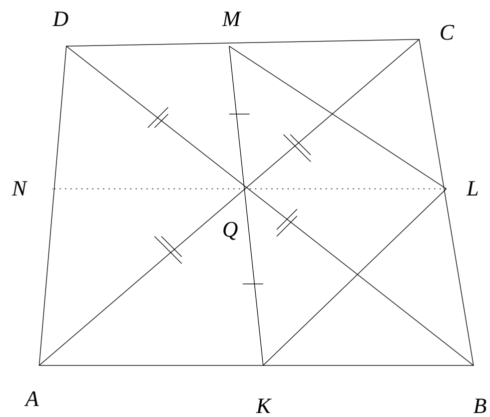

Fig. 1.

Observe that $KLMN$ is a paralellogram, Q is the midpoint of MK and hence NL also passes through Q. Let T be the point of intersection of AC and BD; and let S be the point of intersection of BD and MN.

Consider the triangle MNK. Note that SQ is parallel to NK and Q is the midpoint of MK. Hence S is the mid-point of MN. Since MN is parallel to AC, it follows that T is the mid-point of AC. Now Q is the circumcentre of $\triangle ABC$ and the median BT passes through Q. Here there are two possibilities:

(i) ABC is a right triangle with $\angle ABC = 90°$ and $T = Q$; and

(ii) $T \neq Q$ in which case BT is perpendicular to AC.

Suppose $\angle ABC = 90°$ and $T = Q$. Observe that Q is the circumcentre of the triangle DCB and hence $\angle DCB = 90°$. Similarly $\angle DAB = 90°$. It follows that $\angle ADC = 90°$. and $ABCD$ is a rectangle. This implies that $KLMN$ is a rhombus. Hence $LK/LM = 1$ and this gives $CD = CB$. Thus $ABCD$ is a square.

In the second case, observe that BD is perpendicular to AC, KL is parallel to AC and LM is parallel to BD. Hence it follows that ML is perpendicular to LK. Similar reasoning shows that $KLMN$ is a rectangle.

Using $LK/LM = CD/CB$, we get that CBD is similar to LMK. In particular, $\angle LMK = \angle CBD = \alpha$ say. Since LM is parallel to DB, we also get $\angle BQK = \alpha$. Since $KLMN$ is a cyclic quadrilateral we also get $\angle LNK = \angle LMK = \alpha$. Using the fact that BD is parallel to NK, we get $\angle LQB = \angle LNK = \alpha$. Since BD bisects $\angle CBA$, we also have $\angle KBQ = \alpha$. Thus

$$QK = KB = BL = LQ$$

and BL is parallel to QK. This gives QM is parallel to LC and

$$QM = QL = BL = LC$$

It follows that $QLCM$ is a parallelogram. But $\angle LCM = 90°$. Hence $\angle MQL = 90°$. This implies that $KLMN$ is a square. Also observe that $\angle LQK = 90°$ and hence $\angle CBA = \angle LQK = 90°$. This gives $\angle CDA = 90°$ and hence $ABCD$ is a rectangle. Since $BA = BC$, it follows that $ABCD$ is a square.

2. Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

<u>Solution:</u> We write the equation in the form

$$a^2 + 2ap + p^2 + b(3a + 2p) = 0$$

Hence

$$b = \frac{-(a+p)^2}{3a + 2p}$$

is an integer. This shows that $3a + 2p$ divides $(a + p)^2$ and hence also divides $(3a + 3p)^2$. But, we have

$$(3a + 3p)^2 = (3a + 2p + p)^2 = (3a + 2p)^2 + 2p(3a + 2p) + p^2.$$

It follows that $3a + 2p$ divides p^2. Since p is a prime, the only divisors of p^2 are $\pm 1, \pm p$ and $\pm p^2$. Since $p > 3$, we also have $p = 3k + 1$ or $3k + 2$.

<u>Case 1:</u> Suppose $p = 3k + 1$. Obviously $3a + 2p = 1$ is not possible. Infact, we get $1 = 3a + 2p = 3a + 2(3k + 1) \Rightarrow 3a + 6k = -1$ which is impossible. On the other hand $3a + 2p = -1$ gives $3a = -2p - 1 = -6k - 3 \Rightarrow a = -2k - 1$ and $a + p = -2k - 1 + 3k + 1 = k$.

Thus $b = \dfrac{-(a+p)^2}{(3k + 2p)} = k^2$. Thus $(a, b) = (-2k - 1, k^2)$ when $p = 3k + 1$. Similarly, $3a + 2p = p \Rightarrow 3a = -p$ which is not possible. Considering $3a + 2p = -p$, we get $3a = -3p$ or $a = -p \Rightarrow b = 0$. Hence $(a, b) = (-3k - 1, 0)$ where $p = 3k + 1$.

Let us consider $3a + 2p = p^2$. Hence $3a = p^2 - 2p = p(p - 2)$ and neither p nor $p - 2$ is divisible by 3. If $3a + 2p = -p^2$, then $3a = -p(p + 2) \Rightarrow a = -(3k + 1)(k + 1)$.

Hence $a + p = (3k + 1)(-k - 1 + 1) = -(3k + 1)k$. This gives $b = k^2$. Again $(a, b) = \Big(-(k + 1)(3k + 1), k^2\Big)$ when $p = 3k + 1$.

<u>Case 2:</u> Suppose $p = 3k - 1$. If $3a + 2p = 1$, then $3a = -6k + 3$ or $a = -2k + 1$. We also get

$$b = \frac{-(a+p)^2}{1} = \frac{-(-2k + 1 + 3k - 1)^2}{1} = -k^2$$

and we get the solution $(a,b) = (-2k+1, k^2)$. On the other hand $3a+2p = -1$ does not have any solution integral solution for a. Similarly, there is no solution in the case $3a+2p = p$. Taking $3a+2p = -p$, we get $a = -p$ and hence $b = 0$. We get the solution $(a,b) = (-3k+1, 0)$. If $3a+2p = p^2$, then $3a = p(p-2) = (3k-1)(3k-3)$ giving $a = (3k-1)(k-1)$ and hence $a+p = (3k-1)(1+k-1) = k(3k-1)$. This gives $b = -k^2$ and hence $(a,b) = (3k-1, -k^2)$. Finally $3a+2p = -p^2$ does not have any solution.

3. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a, we denote by $[a]$ the greatest integer not exceeding a.)

Solution: Suppose α is a real root of the given equation. Then

$$\alpha^5 - \alpha^3 + \alpha - 2 = 0. \qquad \cdots(\mathbf{1})$$

This gives $\alpha^5 - \alpha^3 + \alpha - 1 = 1$ and hence $(\alpha-1)(\alpha^4+\alpha^3+1) = 1$. Observe that $\alpha^4+\alpha^3+1 \geq 2\alpha^2+\alpha^3 = \alpha^2(\alpha+2)$. If $-1 \leq \alpha < 0$, then $\alpha+2 > 0$, giving $\alpha^2(\alpha+2) > 0$ and hence $(\alpha-1)(\alpha^4+\alpha^3+1) < 0$. If $\alpha < -1$, then $\alpha^4+\alpha^3 = \alpha^3(\alpha+1) > 0$ and hence $\alpha^4+\alpha^3+1 > 0$. This again gives $(\alpha-1)(\alpha^4+\alpha^3+1) < 0$.

The above resoning shows that for $\alpha < 0$, we have $\alpha^5 - \alpha^3 + \alpha - 1 < 0$ and hence cannot be equal to 1. We conclude that a real root α of $x^5 - x^3 + x - 2 = 0$ is positive (obviously $\alpha \neq 0$).

Now using $\alpha^5 - \alpha^3 + \alpha - 2 = 0$, we get

$$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$$

The statement $[\alpha^6] = 3$ is equivalent to $3 \leq \alpha^6 < 4$.

Consider $\alpha^4 - \alpha^2 + 2\alpha < 4$. Since $\alpha > 0$, this is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$. Using the relation (1), we can write $2\alpha^2 - \alpha + 2 < 4\alpha$ or $2\alpha^2 - 5\alpha + 2 < 0$. Treating this as a quadratic, we get this is equivalent to $\frac{1}{2} < \alpha < 2$. Now observe that if $\alpha \geq 2$ then $1 = (\alpha-1)(\alpha^4+\alpha^3+1) \geq 25$ which is impossible. If $0 < \alpha \leq \frac{1}{2}$, then $1 = (\alpha-1)(\alpha^4+\alpha^3+1) < 0$ which again is impossible. We conlude that $\frac{1}{2} < \alpha < 2$. Similarly $\alpha^4 - \alpha^2 + 2\alpha \geq 3$ is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$ which is equivalent to $2\alpha^2 - 4\alpha + 2 \geq 0$. But this is $2(\alpha-1)^2 \geq 0$ which is valid. Hence $3 \leq \alpha^6 < 4$ and we get $[\alpha^6] = 3$.

4. Let R denote the circumradius of a triangle ABC; a, b, c its sides BC, CA, AB; and r_a, r_b, r_c its exradii opposite A, B, C. If $2R \leq r_a$, prove that

   (i) $a > b$ and $a > c$;

   (ii) $2R > r_b$ and $2R > r_c$.

Solution: We know that $2R = \dfrac{abc}{2\triangle}$ and $r_a = \dfrac{\triangle}{s-a}$, where a, b, c are the sides of the triangle ABC, $s = \dfrac{a+b+c}{2}$ and $\triangle$ is the area of ABC. Thus the given condition $2R \leq r_a$ translates to

$$abc \leq \frac{2\triangle^2}{s-a}$$

Putting $s-a=p, s-b=q, s-c=r$, we get $a=q+r, b=r+p, c=p+q$ and the condition now is

$$p(p+q)(q+r)(r+p) \leq 2\triangle^2$$

But Heron's formula gives, $\triangle^2 = s(s-a)(s-b)(s-c) = pqr(p+q+r)$. We obtain $(p+q)(q+r)(r+p) \leq 2qr(p+q+r)$. Expanding and effecting some cancellations, we get

$$p^2(q+r) + p(q^2+r^2) \leq qr(q+r). \qquad (\star)$$

Suppose $a \leq b$. This implies that $q+r \leq r+p$ and hence $q \leq p$. This implies that $q^2r \leq p^2r$ and $qr^2 \leq pr^2$ giving $qr(q+r) \leq p^2r + pr^2 < p^2r + pr^2 + p^2q + pq^2 = p^2(q+r) + p(q^2+r^2)$ which contradicts $(\star)$. Similarly, $a \leq c$ is also not possible. This proves (i).

Suppose $2R \leq r_b$. As above this takes the form

$$q^2(r+p) + q(r^2+p^2) \leq pr(p+r). \qquad (\star\star)$$

Since $a > b$ and $a > c$, we have $q > p, r > p$. Thus $q^2r > p^2r$ and $qr^2 > pr^2$. Hence

$$q^2(r+p) + q(r^2+p^2) > q^2r + qr^2 > p^2r + pr^2 = pr(p+r)$$

which contradicts $(\star\star)$. Hence $2R > r_b$. Similarly, we can prove that $2R > r_c$. This proves (ii)

5. Let S denote the set of all 6-tuples (a,b,c,d,e,f) of positive integers such that $a^2+b^2+c^2+d^2+e^2 = f^2$. Consider the set

$$T = \Big\{abcdef \; : \; (a,b,c,d,e,f) \in S\Big\}.$$

Find the greatest common divisor of all the members of T.

Solution: We show that the required gcd is 24. Consider an element $(a,d,c,d,e,f) \in S$. We have

$$a^2+b^2+c^2+d^2+e^2 = f^2.$$

We first observe that not all a,b,c,d,e can be odd. Otherwise, we have $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 8$ and hence $f^2 \equiv 5 \pmod 8$, which is impossible because no square can be congruent to 5 modulo 8. Thus at least one of a,b,c,d,e is even.

Similarly if none of a,b,c,d,e is divisible by 3, then $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 3$ and hence $f^2 \equiv 2 \pmod 3$ which again is impossible because no square is congruent to 2 modulo 3. Thus 3 divides $abcdef$.

There are several possibilities for a,b,c,d,e.

Case 1: Suppose one of them is even and the other four are odd; say a is even, b,c,d,e are odd. Then $b^2+c^2+d^2+e^2 \equiv 4 \pmod 8$. If $a^2 \equiv 4 \pmod 8$, then $f^2 \equiv 0 \pmod 8$ and hence $2|a, 4|f$ giving $8|af$. If $a^2 \equiv 0 \pmod 8$, then $f^2 \equiv 4 \pmod 8$ which again gives that $4|a$ and $2|f$ so that $8|af$. It follows that $8|abcdef$ and hence $24|abcdef$.

Case 2: Suppose a,b are even and c,d,e are odd. Then $c^2+d^2+e^2 \equiv 3 \pmod 8$. Since $a^2+b^2 \equiv 0$ or 4 modulo 8, it follows that $f^2 \equiv 3$ or $7 \pmod 8$ which is impossible. Hence this case does not arise.

Case 3: If three of a,b,c,d,e are even and two odd, then $8|abcdef$ and hence $24|abcdef$.

Case 4: If four of a,b,c,d,e are even, then again $8|abcdef$ and $24|abcdef$. Here again for any six tuple (a,b,c,d,e,f) in S, we observe that $24|abcdef$. Since

$$1^2+1^2+1^2+2^2+3^2 = 4^2.$$

We see that (1,1,1,2,3,4)$\in S$ and hence $24 \in T$. Thus 24 is the gcd of T.

6. Prove that the number of 5-tuples of positive integers (a, b, c, d, e) satisfying the equation

$$abcde = 5\big(bcde + acde + abde + abce + abcd\big)$$

is an **odd** integer.

Solution: We write the equation in the form:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{1}{5}.$$

The number of five tuple (a, b, c, d, e) which satisfy the given relation and for which $a \neq b$ is even, because for if (a, b, c, d, e) is a solution, then so is (b, a, c, d, e)which is distinct from (a, b, c, d, e). Similarly the number of five tuples which satisfy the equation and for which $c \neq d$ is also even. Hence it suffices to count only those five tuples (a, b, c, d, e) for which $a = b, c = d$. Thus the equation reduces to

$$\frac{2}{a} + \frac{2}{c} + \frac{1}{e} = \frac{1}{5}.$$

Here again the tuple (a, a, c, c, e) for which $a \neq c$ is even because we can associate different solution (c, c, a, a, e) to this five tuple. Thus it suffices to consider the equation

$$\frac{4}{a} + \frac{1}{e} = \frac{1}{5},$$

and show that the number of pairs (a, e) satisfying this equation is odd.

This reduces to

$$ae = 20e + 5a$$

or

$$(a - 20)(e - 5) = 100.$$

But observe that

$$100 = 1 \times 100 = 2 \times 50 = 4 \times 25 = 5 \times 20$$
$$= 10 \times 10 = 20 \times 5 = 25 \times 4 = 50 \times 2 = 100 \times 1.$$

Note that no factorisation of 100 as product of two negative numbers yield a positive tuple (a, e). Hence we get these 9 solutions. This proves that the total number of five tuples (a, b, c, d, e) satisfying the given equation is odd.